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Superconformal Interpretation of BPS States in AdS Geometries[1]

Sergio Ferrara[2,3] and Emery Sokatchev[4,5]

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We carry out a general analysis of the representations of the superconformal algebras SU(2, $2/N$), OSp(8/4, $\mathbb{R}$), and OSp(8*/$2N$) and give their realization in superspace. We present a construction of their UIRs by multiplication of the different types of massless superfields ("supersingletons"). Particular attention is paid to the so-called "short multiplets." Representations undergoing shortening have "protected dimension" and may correspond to BPS states in the dual supergravity theory in anti-de Sitter space. These results are relevant for the classification of multitrace operators in boundary conformally invariant theories as well as for the classification of AdS black holes preserving different fractions of supersymmetry.

1. INTRODUCTION

The study of superconformal algebras has recently become of central importance because of their dual rôle in describing the gauge symmetries of supergravity in anti-de Sitter bulk and the global symmetries of the boundary field theory (Gubser *et al*., 1998; Maldacena, 1998; Witten, 1998).

A special class of configurations that are particularly relevant are the so-called BPS states, that is, dynamical objects corresponding to representations that undergo "shortening."

These representations can only occur when the conformal dimension of a (super)primary operator is "quantized" in terms of the R symmetry quantum numbers and they are at the basis of the so-called "nonrenormalization" theorems of

[1] Just before submitting this paper to the hep-th archive, we saw a new article by P. Heslop and P. S. Howe (2000). It partially overlaps with our treatment of the $d = 4$ case.

[2] CERN Theoretical Division, CH 1211 Geneva 23, Switzerland.
[3] Department of Physics and Astronomy, University of California, Los Angeles, California 90095.
[4] Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, (UMR 5108 associée à l'Université de Savoie), Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France.
[5] To whom correspondence should be addressed at Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France.

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supersymmetric quantum theories (Ferrara *et al.*, 1974; Iliopoulous and Zumino, 1974; Wess and Zumino, 1974).

There exist different methods of constructing the UIRs of superconformal algebras. One is the so-called oscillator construction of the Hilbert space in which a given UIR acts (Bars and Günaydin, 1983; Duff, 1999; Günaydin, 1982; Günaydin and Hyun, 1988; Günaydin and Marcus, 1985; Günaydin *et al.*, 1998, 1999; Günaydin and Saclioglu, 1982a,b; Nicolai, 1984). Another one, more appropriate to describe field theories, is the realization of such representations on superfields defined in superspaces (Ferrara *et al.*, 1974; Salam and Strathdee, 1974). The latter are "supermanifolds," which can be regarded as the quotient of the conformal supergroup by some of its subgroups.

In the case of ordinary superspace the subgroup in question is the supergroup obtained by exponentiating a nonsemisimple superalgebra, which is the semidirect product of a super-Poincaré graded Lie algebra with dilatation (SO(1, 1)) and the R symmetry algebra. This is the superspace appropriate for non-BPS states. Such states correspond to bulk massive states, which can have "continuous spectrum" of the AdS mass (or, equivalently, of the conformal dimension of the primary fields).

BPS states are naturally associated to superspaces with lower number of "odd" coordinates and, in most cases, with some internal coordinates of a coset space G/H. Here G is the R symmetry group of the superconformal algebra, that is, the subalgebra of the even part that commutes with the conformal algebra of space-time, and H is some subgroup of G having the same rank as G.

Such superspaces are called "harmonic" (Galperin *et al.*, 1984) and they are characterized as having a subset of the initial odd coordinates θ. The complementary number of odd variables determines the fraction of supersymmetry preserved by the BPS state. If a BPS state preserves K supersymmetries then the θs of the associated harmonic superspace will transform under some UIR of H_K.

For $1/2$ BPS states, that is, states with maximal supersymmetry, the superspace involves the minimal number of odd coordinates (half of the original one) and H_K is then a maximal subgroup of G. On the other hand, for states with the minimal fraction of supersymmetry H_K reduces to the "maximal torus" whose Lie algebra is the Cartan subalgebra of G.

It is the aim of this paper to give a comprehensive treatment of BPS states related to "short representations" of superconformal algebras for the cases that are most relevant in the context of the AdS/CFT correspondence, that is, for $d = 3\,(N = 8)$, $d = 6$, and $d = 4$ (for arbitrary N). The underlying conformal field theories correspond to world-volume theories of N_c copies of M_2, M_5, and D_3 branes in the large N_c limit (Aharony *et al.*, 1998; Aharony *et al.*, 1998; Claus *et al.*, 1998; Ferrara *et al.*, 1998; Halyo, 1998; Leigh and Rozsli, 1998; Minwalla, 1998) that are "dual" to AdS supergravities describing the horizon geometry of the branes (Aharony *et al.*, 2000).

Some of the results presented in this paper have already appeared elsewhere (Andrianopoli, 1998; Ferrara and Sokatchev, 1999, 2000, 2000).[6] Here we give a systematic and unified treatment of the BPS states corresponding to the three superconformal algebras discussed previously. The method we use in developed in full detail in the case of $d = 4$ superconformal algebra SU(2, 2/N) in Sections 2–5. In Section 2 we carry out an abstract analysis of the conditions for Grassmann (G) analyticity (Galperin *et al.*, 1981) (the generalization of the familiar concept of chirality (Ferrara *et al.*, 1974)) in a superconformal context. We find the constraints on the conformal dimension and R symmetry quantum numbers of a superfield following from the requirement that it does not depend on one or more Grassmann variables. Introducing G analyticity in a traditional superspace cannot be done without breaking the R symmetry. The latter can be restored by extending the superspace by harmonic variables (Bandos, 1988; Galperin *et al.*, 1984; Galperin *et al.*, 1985; Galperin *et al.*, 1987; Ivanov *et al.*, 1985; Kallosh, 1985; Rosly, 1983, 1985; Rosly and Schwarz, 1986) parametrizing the coset G/H_K. In Section 3 the (N, p, q) harmonic superspaces (Galperin *et al.*, 1987; Hartwell and Howe, 1995a,b) relevant to the description of BPS states preserving $p + q/2N$ supersymmetries are reviewed. In Section 4 the massless UIRs ("supersingleton" multiplets) (Binegar, 1986; Dobrev and Petkova, 1985, 1987; Flato and Fronsdal, 1978, 1980, 1981, 1986; Fronsdal, 1982) of SU(2, 2/N) are considered, first as constrained superfields in ordinary superspace (Howe *et al.*, 1981; Siegel, 1981) and then, for a part of them, as $(N, p, N - p)$ G-analytic harmonic superfields (Galperin *et al.*, 1984; Hartwell and Howe, 1995a,b). In Section 5 we use supersingleton multiplication to construct UIRs of SU(2, 2/N). We show that in this way one can reproduce the complete classification of UIRs of Dobrev and Petkova (1985, 1987). We give the full list of BPS states obtained by multiplying chiral and G-analytic supersingletons as well as the restricted classes of BPS states obtained from one type of G-analytic supersingleton alone. We also discuss different kinds of shortening that certain superfields (not of the BPS type) may undergo. In Sections 6 and 7 we apply the same method to extend these results to $d = 6$ and $d = 3$ for the superalgebras of the maximal supersymmetries, that is, OSp($8^*/2N$) and OSp(8/4, $\mathbb{R}$). We conclude the paper by listing the various BPS states in the physically relevant cases of D3, M_2, and M_5 branes horizon geometry where only one type of supersingletons appears.

Applications of the present results are found (Andrianopoli *et al.*, 1999; Ferrara and Zaffaroni, 1999) in the classification of multitrace operators in four-dimensional $N = 4$ SU(N_c) Yang–Mills theory (Bianchi and Kovacs, 1999; Bianchi *et al.*, 1999; Chalmers and Schalm, 1999; D'Hoker *et al.*, 1999; D'Hoker *et al.*, 1999; Skiba, 1999), dual to type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ (Maldacena, 1998, 1999).

[6] The new results were reported by one of us at the Workshop on "Strings, Branes and M-theory" at the CIT-USC Center for Theoretical Physics, Los Angeles, California on April 5 and 7, 2000.

Another area of interest is the classification of AdS black holes (Behrndt, *et al.*, 1998; Chamblin *et al.*, 1999; Duff and Liu, 1999; Hawking *et al.*, 1999) according to the fraction of supersymmetry preserved by the black hole background.

In a parallel analysis with black holes in asymptotically flat background (Ferrara and Günaydin, 1998; Ferrara and Maldacena, 1998), the AdS/CFT correspondence predicts that such BPS states should be dual to superconformal states undergoing "shortening" of the type discussed here.

2. GRASSMANN ANALYTICITY AND CONFORMAL SUPERSYMMETRY

In this section we shall study the realizations of $D = 4$ N-extended conformal supersymmetry SU(2, 2/N) on superfields depending on a subset of the $4N$ odd variables. Such superfields will be called G analytic.

The nonvanishing (anti)commutation relations involving the odd generators of the superalgebra SU(2, 2/N) are given below:

$$\begin{aligned}
\left\{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\right\} &= 2\delta^i_j(\sigma^\mu)_{\alpha\dot\alpha}P_\mu,\\
\left\{S_{\alpha j}, \bar{S}^i_{\dot\alpha}\right\} &= 2\delta^i_j(\sigma^\mu)_{\alpha\dot\alpha}K_\mu,\\
\left\{Q^i_\alpha, S^\beta_j\right\} &= -\delta^i_j(\sigma^{\mu\nu})^\beta_\alpha M_{\mu\nu} - 4\delta^\beta_\alpha T^i_j - 2\delta^\beta_\alpha\delta^i_j(R + iD),\\
\left[Q^i_\alpha, K_\mu\right] &= -(\sigma_\mu)_{\alpha\dot\alpha}\bar{S}^{\dot\alpha i}, \qquad [\bar{Q}_{\dot\alpha i}, K_\mu] = (\sigma_\mu)_{\alpha\dot\alpha}S^\alpha_i,\\
[S_{\alpha i}, P_\mu] &= -(\sigma_\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}_i, \qquad \left[\bar{S}^i_{\dot\alpha}, P_\mu\right] = (\sigma_\mu)_{\alpha\dot\alpha}Q^{\alpha i},
\end{aligned} \tag{2.1}$$

Here the odd generators are[7]: Q^i_α, $\bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^\dagger$ of Poincaré supersymmetry and $S_{\alpha i}$, $\bar{S}^i_{\dot\alpha} = (S_{\alpha i})^\dagger$ of special conformal supersymmetry. The even generators are as follows: P_μ of translations, K_μ of conformal boosts, $M_{\mu\nu} = -M_{\nu\mu}$ of the Lorentz group, D of dilatations, T^i_j of SU(N), and R of $U(1)$ ("R charge").

Further, the Lorentz and SU(N) generators commute with Q as follows:

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})^\beta_\alpha Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = \frac{1}{2}(\tilde{\sigma}_{\mu\nu})^{\dot\beta}_{\dot\alpha}\bar{Q}_{\dot\beta}, \tag{2.2}$$

$$\left[T^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{N}\delta^i_j Q^k, \qquad \left[T^i_j, \bar{Q}_k\right] = -\delta^i_k\bar{Q}_j + \frac{1}{N}\delta^i_j\bar{Q}_k, \tag{2.3}$$

and similarly for S. Next, the commutators of Q and S with the dilatation and R charge generators are given below:

$$[D, Q] = \frac{i}{2}Q, \qquad [D, \bar{Q}] = \frac{i}{2}\bar{Q};$$

[7] Two-component spinor indices are raised and lowered with the help of the Levi–Civita tensor: $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\bar{\chi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\bar{\chi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{\chi}^{\dot\beta}$; $\epsilon_{12} = \epsilon_{\dot1\dot2} = -\epsilon^{12} = -\epsilon^{\dot1\dot2} = 1$.

$$[D, S] = -\frac{i}{2}S, \qquad [D, \bar{S}] = -\frac{i}{2}\bar{S}; \tag{2.4}$$

$$[R, Q] = \frac{4-N}{2N}Q, \qquad [R, \bar{Q}] = -\frac{4-N}{2N}\bar{Q};$$
$$[R, S] = -\frac{4-N}{2N}S, \qquad [R, \bar{S}] = \frac{4-N}{2N}\bar{S}. \tag{2.5}$$

Finally, the SU(N) generators T^i_j, $(T^i_j)^\dagger$, $\sum_{i=1}^N T^i_i = 0$ form the algebra

$$\left[T^i_j, T^k_l\right] = \delta^k_j T^i_l - \delta^i_l T^k_j. \tag{2.6}$$

The rest of the superalgebra SU(2, 2/N) is the conformal algebra of M, P, K, D, which will not be needed here.

The superspace traditionally used for the realization of SU(2, 2/N) (as well as for Poincaré supersymmetry) is given by the real coset

$$\mathbb{R}^{4|2N,2N} = \frac{\mathrm{SU}(2, 2/N)}{\{K, S, \bar{S}, M, D, T, R\}} = \left(x^\mu, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}i}\right). \tag{2.7}$$

It is parametrized by four even coordinates x^μ and $2N$ left-handed odd spinor coordinates θ^α_i in the fundamental of SU(N) together with the $2N$ right-handed complex conjugates $\bar{\theta}^{\dot{\alpha}i} = \overline{\theta^\alpha_i}$. The superalgebra is realized on superfields $\Phi(x, \theta, \bar{\theta})$ defined as functions in the coset (2.7). The generators of the coset denominator K, S, $\bar{S}$, M, D, T, R act on the superspace coordinates as well as on the external indices of the superfield. The latter action is given by the matrix parts of these generators, $K_\mu \to k_\mu$, $S_{\alpha i} \to s_{\alpha i}$, $\bar{S}^i_\alpha \to \bar{s}^i_\alpha$, $M_{\mu\nu} \to m_{\mu\nu}$, $D \to i\ell$, $T^i_j \to t^i_j$, $R \to r$.[8] According to the definition of a (super)conformal primary field, the matrix parts of the transitive generators K and S vanish:

$$s_{\alpha i}\Phi = \bar{s}^i_{\dot{\alpha}}\Phi = k_\mu\Phi = 0 \tag{2.8}$$

(the third constraint follows from the first two, see (2.1)). The homogeneous action of the remaining ones, d, l, r, t, on the superfield and, in particular, on its lowest component $\phi(x) = \Phi|_{\theta=\bar{\theta}=0}$ defines the latter as an irrep of SO(1, 1) $\times$ SL(2, $\mathbb{C}$) $\times$ U(1) $\times$ SU(N) with the following quantum numbers:

$$D(\ell; j_1, j_2; r; a_1, \ldots, a_{N-1}) \tag{2.9}$$

where ℓ is the conformal dimension, j_1 and j_2 are the two Lorentz quantum numbers ("spins"), r is the R charge and $a_1, \ldots, a_{N-1}$ are the SU(N) Dynkin labels.

[8] We assign the R charge $r_\theta = -(4-N)/2N$ to the left-handed Grassmann coordinates θ^α in order to be consistent with the convention that chiral superfields $\Phi(\theta)$ have $r = -\ell$ for any N (see (2.13)). Note that for $N = 4$, $r_\theta = 0$ and the r quantum number becomes a "central charge" (Binegar, 1986; Dobrev and Petkova, 1985, 1987). In this case one refers to the PSU(2, 2/4) algebra for $r = 0$ and to the PU(2, 2/4) algebra for $r \neq 0$.

2.1. Chiral Superfields

The superalgebra SU(2, 2/N) can be realized in a smaller superspace, called "chiral" superspace. It is obtained by adding half of the Poincaré supersymmetry generators, for instance, the right-handed ones, $\bar{Q}_i^{\dot\alpha}$, to the coset denominator:

$$\mathbb{C}^{4|2N,0} = \frac{\mathrm{SU}(2,2/N)}{\{K, S, S, M, D, T, R, \bar{Q}\}} = \left(x^\mu, \theta_i^\alpha\right). \tag{2.10}$$

This means adding a new constraint to the set (2.8):

$$\bar{q}_i^{\dot\alpha}\Phi = 0 \tag{2.11}$$

where $\bar{q}$ is the matrix part of the generator $\bar{Q}$. However, in this case the superalgebra (2.1) implies restrictions on the allowed values of the quantum numbers (2.9) (Wess and Zumino, 1974b). Indeed, the constraints (2.11) and (2.8) yield the compatibility condition

$$\left\{\bar{q}_i^{\dot\alpha}, \bar{s}_{\dot\beta}^j\right\}\Phi = \left[-\delta_i^j(\sigma^{\mu\nu})_{\dot\beta}^{\dot\alpha} m_{\mu\nu} - 2\delta_{\dot\beta}^{\dot\alpha}\left(2t_j^i + \delta_i^j(\ell + r)\right)\right]\Phi = 0. \tag{2.12}$$

This is only possible if the superfield (i.e., its first component (2.9)) carries no right-handed spin, no SU(N) indices and has R charge $r = -\ell$:

$$\mathbb{C}^{4|2N,0} \Rightarrow \mathcal{D}(\ell; j_1, 0; -\ell; 0, \ldots, 0). \tag{2.13}$$

Such superfields are called (left-handed) chiral. Note that both the superspace (2.10) and the superfields defined in it are complex.

Given a general superfield $\Phi(x, \theta, \bar\theta)$, one can restrict it to the coset (2.10) by imposing the following differential "chirality" constraint (Ferrara *et al.*, 1974)

$$\bar{D}_i^{\dot\alpha}\Phi(x, \theta, \bar\theta) = 0. \tag{2.14}$$

Here $\bar{D}$ is the right-handed half of the "covariant spinor derivatives"

$$D_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} + i\bar\theta^{\dot\alpha i}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu, \qquad \bar{D}_{\dot\alpha i} = -\frac{\partial}{\partial\theta^{\dot\alpha i}} - i\theta_i^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu. \tag{2.15}$$

Note that these derivatives are only covariant with respect to the super-Poincaré subalgebra of SU(2, 2/N). They obey the following anticommutation relations:

$$\left\{D_\alpha^i, D_\beta^j\right\} = \{\bar{D}_{\dot\alpha i}, \bar{D}_{\dot\beta j}\} = 0, \qquad \left\{D_\alpha^i, \bar{D}_{\dot\beta j}\right\} = -2i\delta_j^i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu. \tag{2.16}$$

A crucial observation is that the chirality constraint (2.14) can be solved by going to the "left-handed chiral" basis

$$x_L^\mu = x^\mu + i\theta_{Li}\sigma^\mu\bar\theta_L^i, \qquad \theta_{Li}^\alpha = \theta_i^\alpha, \qquad \bar\theta_L^{\dot\alpha i} = \bar\theta^{\dot\alpha i}. \tag{2.17}$$

There $\bar{D}$ becomes just a partial derivative, $\bar{D}_{\dot\alpha i} = -\partial/\partial\bar\theta_L^{\dot\alpha i}$, so (2.14) simply implies

$$\Phi = \Phi\left(x_L^\mu, \theta_{Li}^\alpha\right). \tag{2.18}$$

An important property of the chiral superfields (2.18) is that the product of two of them is still a chiral superfield, that is, they form a "ring structure." Note the close analogy with the typical property of ordinary analytic functions. As we shall see in the next subsection, this analogy can be further developed.

2.2. Grassmann Analytic Superfields

A natural question is whether or not one can find other realizations of SU(2, 2/N) in superspaces involving only part of the odd coordinates. In the chiral case discussed previously, we chose to add all of the right-handed generators $\bar{Q}_i^{\dot{\alpha}}$, which form an irrep of SU(N), to the coset denominator. Now, let us assume for a moment the possibility to break SU(N).[9] We can then take just one of the Qs or the $\bar{Q}$s, for example, Q_α^1 and put it in the denominator. The resulting coset has $2N-2$ left-handed and $2N$ right-handed odd coordinates:

$$\mathbb{A}^{4|2N-2,2N} = \frac{\mathrm{SU}(2,2/N)}{\{K, S, S, M, D, T, R, Q^1\}} = \left(x^\mu, \theta_2^\alpha, \ldots, \theta_N^\alpha, \bar{\theta}_{\dot{\alpha}}^1, \ldots, \bar{\theta}_{\dot{\alpha}}^N\right) \tag{2.19}$$

This means replacing the chirality condition (2.11) by

$$q_\alpha^1 \Phi = 0. \tag{2.20}$$

Then, a compatibility condition analogous to (2.12) follows from the anticommutator

$$\left\{q_\alpha^1, s_1^\beta\right\}\Phi = \left[-(\sigma^{\mu\nu})_\alpha^\beta m_{\mu\nu} - 2\delta_\alpha^\beta\left(2t_1^1 - \ell + r\right)\right]\Phi = 0. \tag{2.21}$$

It implies $(\sigma^{\mu\nu})_\alpha{}^\beta m_{\mu\nu}\Phi = 0$, that is, no left-handed spin, as well as a relation between the eigenvalue of the SU(N) generator t_1^1, the R charge and the conformal dimension:

$$j_1 = 0, \qquad 2t_1^1 = \ell - r. \tag{2.22}$$

Further, anticommuting q_α^1 with the remaining projections $s_{2,3,\ldots,N}^\beta$, we obtain

$$t_i^1 = 0, \quad 2 \le i \le N. \tag{2.23}$$

Let us now make a digression and discuss the SU(N) generators t_j^i. In the Cartan decomposition of the SU(N) algebra (2.6) the generators with $1 \le i < j \le N$ are associated to the positive roots ("raising operators"). Among them t_{i+1}^i, $i = 1, \ldots, N-1$ correspond to the simple roots, which means that the other raising operators are obtained by commuting the simple ones. Similarly, the generators with $N \ge i > j \ge 1$ are associated to the negative roots ("lowering operators"), the simple ones being t_i^{i+1}, $i = 1, \ldots, N-1$. Finally, the $N-1$ independent

[9] Superspaces of this type can be introduced without breaking SU(N) in the framework of harmonic superspace, see Section 3.

generators t_i^i (recall that $\sum_{i=1}^N t_i^i = 0$) define the $N-1$ charges of the Cartan subalgebra of $[\mathrm{U}(1)]^{N-1} \subset \mathrm{SU}(N)$ as follows:

$$m_k = t_k^k - t_N^N = t_k^k + \frac{m}{N}, \quad 1 \le k \le N, \; m = \sum_{i=1}^N m_i \tag{2.24}$$

where $m_N \equiv 0$. An irrep of $\mathrm{SU}(N)$ is generated from the highest weight state (HWS) $|a_1, \ldots, a_{N-1}\rangle$ specified, for example, by the Dynkin labels defined by

$$a_k = m_k - m_{k+1} \ge 0, \quad 1 \le k \le N-1. \tag{2.25}$$

Correspondingly, the charges (2.24) of a HWS take eigenvalues $m_1 \ge m_2 \ge \ldots$ $m_{N-1} \ge m_N = 0$. In the language of Young, tableaux m_k is just the number of boxes in the kth row. The HWS is by definition annihilated by all the raising operators:

$$t_j^i |a_1, \ldots, a_{N-1}\rangle = 0, \quad 1 \le i < j \le N. \tag{2.26}$$

In these terms, conditions (2.23) are just a subset of the irreducibility conditions (2.26). From (2.22) we obtain the following restrictions on the quantum numbers:

$$\frac{2m}{N} - 2m_1 = r - \ell. \tag{2.27}$$

We can go on and consider a superspace of the type (2.19) where the first $p\,\theta$s are missing:

$$\begin{aligned} \mathbb{A}^{4|2N-2p,2N} &= \frac{\mathrm{SU}(2,2/N)}{\{K, S, S, M, D, T, R, Q^1, \ldots, Q^p\}} \\ &= \left(x^\mu, \theta_{p+1}^\alpha, \ldots, \theta_N^\alpha, \bar\theta_{\dot\alpha}^1, \ldots, \bar\theta_{\dot\alpha}^N\right). \end{aligned} \tag{2.28}$$

As before, this means to impose

$$q_\alpha^i \Phi = 0, \quad 1 \le i \le p. \tag{2.29}$$

Then, from the anticommutators $\{q_\alpha^i, s_i^\beta\} = 0, 1 \le i \le p$ we obtain conditions similar to (2.27):

$$\frac{2m}{N} - 2m_i = r - \ell, \quad 1 \le i \le p. \tag{2.30}$$

Also, $\{q_\alpha^i, s_j^\beta\} = 0$ for $1 \le i < j \le p$ yields a bigger subset of the irreducibility conditions (2.26). In addition, this time we obtain a new type of condition:

$$t_j^i |a_1, \ldots, a_{N-1}\rangle = 0, \quad p \ge i > j \ge 1. \tag{2.31}$$

The generators in (2.31) are lowering operators of SU(N). In fact, these new constraints are corollaries of (2.30). Indeed, from (2.30) follows

$$a_1 = \cdots = a_{p-1} = 0 \quad \text{for } p \geq 2. \tag{2.32}$$

Now, the HWS $|a_1, \ldots, a_{N-1}\rangle$ has the property[10]

$$\left(t_k^{k+1}\right)^{a_k+1} |a_1, \ldots, a_{N-1}\rangle = 0. \tag{2.33}$$

Then it is obvious that (2.32) and (2.33) imply (2.31).

The argument above can be reversed. Take a superfield defined in the superspace $\mathbb{A}^{4|2N-2,2N}$ (2.19) whose lowest component is in the SU(N) irrep with Dynkin labels $[0, \ldots, 0, a_p, \ldots, a_{N-1}]$, $p > 1$. Then (2.31) holds and combining it with the constraint (2.20), we obtain the full set of constraints (2.29). Thus, such a superfield effectively lives in a smaller superspace.

It is clear that we can repeat the same procedure in the right-handed sector. This time the starting point will be a superspace where $\bar{\theta}_{\dot{\alpha}}^N$ is absent (note that in our convention q^1 and $\bar{q}_N$ are the HWSs of the fundamental irrep of SU(N) and of its conjugate, respectively). From the corresponding condition $\bar{q}_N^{\dot{\alpha}} \Phi = 0$ we derive

$$j_2 = 0, \qquad \frac{2m}{N} = \ell + r. \tag{2.34}$$

Going on and removing q right-handed odd variables, $\bar{\theta}_{\dot{\alpha}}^N, \ldots, \bar{\theta}_{\dot{\alpha}}^{N-q+1}$, that is, imposing the constraints

$$\bar{q}_i^{\dot{\alpha}} \Phi = 0, \quad N - q + 1 \leq i \leq N, \tag{2.35}$$

in addition to (2.34) we find

$$m_i = 0, \quad N - q + 1 \leq i \leq N - 1 \ \text{ for } q \geq 2. \tag{2.36}$$

As before, this implies the vanishing of the last $q - 1$ Dynkin labels:

$$a_i = 0, \quad N - q + 1 \leq i \leq N - 1 \ \text{ for } q \geq 2. \tag{2.37}$$

Correspondingly, the HWS is annihilated by the lowering operators t_j^i, $N \geq i > j \geq N - q + 1$.

Finally, we can combine left- and right-handed constraints and define the most general G-analytic superspace as follows:

$$\begin{aligned} \mathbb{A}^{4|2N-2p,2N-2q} &= \frac{\mathrm{SU}(2, 2/N)}{\{K, S, S, M, D, T, R, Q^1, \ldots, Q^p, Q_{N-q+1}, \ldots, Q_N\}} \\ &= \left(x^\mu, \theta_{p+1}^\alpha, \ldots, \theta_N^\alpha, \bar{\theta}_{\dot{\alpha}}^1, \ldots, \bar{\theta}_{\dot{\alpha}}^{N-q}\right), \quad p + q \leq N. \end{aligned} \tag{2.38}$$

[10] The explanation is as follows. The generators t_k^{k+1}, t_{k+1}^k, and $t_k^k - t_{k+1}^{k+1}$ form the algebra of SU(2)$_k \subset$ SU(N). The state $|a_1, \ldots, a_{N-1}\rangle$ can be regarded as the HWS of an irrep of this SU(2)$_k$ of U(1) charge a_k, that is of dimension $a_k + 1$. Equation (2.33) then follows from the fact that t_k^{k+1} is the lowering operator of SU(2)$_k$.

Following Hartwell and Howe (1995a,b) we shall call (2.38) an "(N, p, q) superspace."[11] It is important to realize that anticommuting the Qs and Qs in the denominator should not produce the translation generator P_μ that belongs to the coset. This explains the condition $p + q \leq N$ in (2.38). The superfields defined in this coset are annihilated by a subset of the Poincaré supersymmetry generators:

$$q^i_\alpha \Phi = \bar{q}^{\dot\alpha}_j \Phi = 0, \quad 1 \leq i \leq p, \quad N - q + 1 \leq j \leq N. \tag{2.39}$$

These conditions lead to restrictions on the quantum numbers obtained by combining the ones found above:

$$\begin{aligned} j_1 &= j_2 = 0; \\ \ell &= m_1; \\ r &= \frac{2m}{N} - m_1; \\ m_1 &= m_2 = \cdots = m_p, \\ m_i &= 0, \quad N - q + 1 \leq i \leq N - 1, \quad q \geq 2. \end{aligned} \tag{2.40}$$

Such SU(N) representations have the first $p - 1$ and the last $q - 1$ Dynkin labels vanishing:

$$[0, \ldots, 0, a_p, \ldots, a_{N-q}, 0, \ldots, 0]. \tag{2.41}$$

An interesting limiting case is obtained when $p + q = N$. Such superspaces contain exactly one half of the initial number of Grassmann variables (p left-handed and $N - p$ right-handed spinors). The SU(N) representation of the lowest component of the superfield has only one nonvanishing Dynkin label, $a_p \neq 0$. Consequently, $\ell = a_p$ and $r = (\frac{2p}{N} - 1)\, a_p$. In Section 4 we shall see that in the special case $a_p = 1$ such superfields describe some of the massless superconformal multiplets.

We remark that chiral superspace can be viewed as a limiting case of the above when, for example, $p = 0$ and $q = N$. In this case only $j_1 = 0$, the other Lorentz quantum number j_2 remains arbitrary.

3. (N, p, q) HARMONIC SUPERSPACE

The chiral superspace introduced in Section 2.1 is naturally realized in terms of superfields satisfying a differential constraint of the type (2.14). The question arises if we can formulate similar differential constraints restricting a superfield to the G-analytic superspaces of Section 2.2. It is quite clear that one should impose constraints similar to (2.39) with the supersymmetry generators replaced by spinor covariant derivatives. The only problem is that in (2.29) we have explicitly broken

[11] The first example of a (3, 2, 1) superspace was given in Galperin *et al.* (1987).

the SU(N) invariance, just like when the concept of G analyticity ($N = 2$) was first introduced in Galperin *et al.* (1981). This can be repaired by extending the framework of standard superspace to the so-called harmonic superspace (Galperin *et al.*, 1984).

3.1. Harmonic Variables on the Coset SU(N)/[U(1)]$^{N-1}$

Harmonic superspace is obtained from the ordinary one (2.7) by tensoring it with a coset of the group SU(N)/H where H is a maximal subgroup of SU(N). To be able to describe the most general case of G-analytic superfields one has to choose the smallest such subgroup, which is the Cartan subgroup [U(1)]$^{N-1}$. The resulting coset SU(N)/[U(1)]$^{N-1}$ (introduced in Galperin *et al.* (1984) for $N = 2$, in Galperin *et al.* (1985); Kallosh (1985); Rosly and Schwarz (1986) for $N = 3$ and in Bandos (1988) for arbitrary N) is a compact complex manifold ("flag manifold" (Knapp, 1977, 1986; Hartwell and Howe, 1995a,b)) of complex dimension $N(N-1)/2$. Note, however, that (N, p, q) superfields for $p \geq 2$ and/or $q \geq 2$ effectively live in the smaller cosets SU(N)/[U(1)]$^{N-p-q+1} \times$ SU(p) $\times$ SU(q), as we shall explain later (see also Hartwell and Howe (1995a,b)).

3.1.1. Covariant Description of the Coset SU(N)/[U(1)]$^{N-1}$

The harmonic variables u_i^I and their conjugates $u_I^i = (u_i^I)^*$ form an SU(N) matrix where i is an index in the fundamental representation of SU(N) and $I = 1, \ldots, N$ are the projections of the second index onto the subgroup [U(1)]$^{N-1}$. Further, we define two *independent* SU(N) groups, a left one acting on the index i and a right one acting on the projected index I of the harmonics:

$$\left(u_i^I\right)' = \Lambda_i^j u_j^J \Sigma_J^I, \quad \Lambda \in \mathrm{SU}(N)_L, \ \Sigma \in \mathrm{SU}(N)_R. \tag{3.1}$$

In particular, the charge operators (2.24) of SU(N)$_R$ act on the harmonics as follows:

$$m_K\, u_i^I = (\delta_{KI} - \delta_{KN}) u_i^I, \qquad m_K u_I^i = -(\delta_{KI} - \delta_{KN}) u_I^i. \tag{3.2}$$

The harmonics satisfy the following SU(N) defining conditions:

$$\begin{aligned} & u_i^I u_J^i = \delta_J^I, \\ u \in \mathrm{SU}(N)\colon\ & u_i^I u_I^j = \delta_i^j, \\ & \varepsilon^{i_1 \cdots i_N} u_{i_1}^1 \cdots u_{i_N}^N = 1. \end{aligned} \tag{3.3}$$

3.1.2. Harmonic Functions

A basic assumption of the harmonic approach to the coset $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-1}$ is that any harmonic function is homogeneous under the action of $[\mathrm{U}(1)_R]^{N-1}$, that is, it is an eigenfunction of the charge operators m_I,

$$m_I\, f^{K_1\cdots K_q}_{L_1\cdots L_r}(u) = (\delta_{K_1 I} - \delta_{K_1 N} - \delta_{L_1 I} + \delta_{L_1 N} + \cdots)\, f^{K_1\cdots K_q}_{L_1\cdots L_r}(u) \tag{3.4}$$

(note that the projections (charges) $K_1\cdots K_q; L_1\cdots L_r$ are not necessarily all different). Thus the harmonic function effectively depends on the $(N^2-1)-(N-1) = N(N-1)$ real coordinates of the coset $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-1}$. This description of the coset is global and coordinateless. The function (3.4) is given by its harmonic expansion on the coset (hence the term "harmonic space"). In our $\mathrm{SU}(N)$ covariant notation this expansion is $[\mathrm{U}(1)_R]^{N-1}$ *covariant* and $\mathrm{SU}(N)_L$ *invariant*. To give a simple example, consider the case $N = 2$ and the harmonic function

$$\begin{aligned} f^1(u) = {} & f^i u^1_i + f^{ijk} u^1_i u^1_j u^2_k + \cdots \\ & + f^{i_1\cdots i_{n+1} j_1\cdots j_n} u^1_{i_1}\cdots u^1_{i_{n+1}} u^2_{j_1}\cdots u^2_{j_n} + \cdots . \end{aligned} \tag{3.5}$$

Note that each term in the expansion has the same overall $\mathrm{U}(1)_R$ charge 1. The first coefficient f^i is in the fundamental of $\mathrm{SU}(2)_L$, and the following ones are symmetric in all of their indices (either because $u^1_i u^1_j$ is symmetric in i and j or because the antisymmetrization of $u^1_i u^2_j$ reduces it to a preceding term in (3.5)), thus realizing irreps of $\mathrm{SU}(2)_L$ of isospin $n + 1/2$. As a second example, consider the function

$$f^1_2(u) \equiv f^{11} f^{ij} u^1_i u^1_j + f^{ijkl} u^1_i u^1_j u^1_k u^2_l + \cdots . \tag{3.6}$$

This time the overall charge is even, therefore the irreps of the expansion carry integer isospin.

We remark that the irreducible products of harmonics play the rôle of the familiar spherical harmonics in the case $N = 2$, where the coset $\mathrm{SU}(2)/\mathrm{U}(1) \sim S^2$ (see Galperin *et al.* (1984) for details).

The above $N = 2$ examples are generalized to any N as follows.[12] Consider first a function of the type

$$f^{\underbrace{1\cdots 1}_{m_1}\underbrace{2\cdots 2}_{m_2}\cdots\underbrace{N-1\cdots N-1}_{m_{N-1}}}(u), \quad m_1 \geq m_2 \geq \ldots \geq m_{N-1}. \tag{3.7}$$

Note that the charges form a sequence corresponding to the canonical structure of a Young tableau. This tableau defines the smallest irrep of $\mathrm{SU}(N)_L$ that one finds in the expansion. All the remaining irreps are obtained by the following procedure. Denote the HWS of the smallest irrep by its Dynkin labels, $|a_1, \ldots, a_{N-1}\rangle$ and that of any irrep present in the expansion by $|A_1, \ldots, A_{N-1}\rangle$. The vector $|a_1, \ldots, a_{N-1}\rangle$

[12] We are grateful to P. Sorba for help in developing this argument.

appears in the multiplet generated by the HWS $|A_1, \ldots, A_{N-1}\rangle$ so it can be obtained by the action of the lowering operators of $\mathrm{SU}(N)_L$:

$$|a_1, \ldots, a_{N-1}\rangle = \left(t_1^2\right)^{n_1}\left(t_2^3\right)^{n_2}\cdots\left(t_{N-1}^N\right)^{n_{N-1}}\left|A_1, \ldots, A_{N-1}\right\rangle. \tag{3.8}$$

Here we only use the simple roots; the ordering in (3.8) is of no importance for our argument. From the $\mathrm{SU}(N)$ algebra we easily find the following relations between the two sets of Dynkin labels:

$$A_k = a_k + 2n_k - n_{k-1} - n_{k+1} \geq 0, \quad k = 1, \ldots, N-1. \tag{3.9}$$

Note that the coefficients in (3.9) form the Cartan matrix of $\mathrm{SU}(N)$. The total number of boxes of the Young tableaux (i.e., number of indices of the coefficients, see later) is given by

$$M = \sum_{k=1}^{N-1} kA_k = m + Nn_{N-1}. \tag{3.10}$$

Thus one finds an $N-1$-parameter family of irreps where the choice of the parameters n_k is limited only by the requirements $A_k \geq 0$.

As an illustration of the preceding discussion, look at the first term in the expansion of the function (3.7):

$$f^{i_1\cdots i_{m_1} j_1\cdots j_{m_2}\cdots k_1\cdots k_{m_{N-1}}} u^1_{i_1}\cdots u^1_{i_{m_1}} u^2_{j_1}\cdots u^2_{j_{m_2}}\cdots u^{N-1}_{k_1}\cdots u^{N-1}_{k_{m_{N-1}}}. \tag{3.11}$$

Unlike the simple SU(2) examples above, here the coefficients f are not necessarily irreducible under $\mathrm{SU}(N)_L$. Indeed, they only possess the symmetry associated with each type of harmonic projection, but no antisymmetrization between any two different projections has been performed. Comparing the term (3.11) to the general case (3.9) we can say that in (3.11) the total number of indices (boxes in a Young tableau) is $M = m$, so what is left is the $N-2$-parameter family of irreps corresponding to $n_{N-1} = 0$.

The general term in the expansion of the function (3.7) is obtained from (3.11) by multiplying it by the chargeless harmonic monomial $u^1_{i_1}\cdots u^N_{i_N}$ (the total antisymmetrization of the indices $i_1, \ldots, i_N$ results in an $\mathrm{SU}(N)_L$ singlet, so it should be eliminated):

$$\begin{aligned} &f^{\underbrace{1\cdots 1}_{m_1}\underbrace{2\cdots 2}_{m_2}\cdots\underbrace{N-1\cdots N-1}_{m_{N-1}}}(u) \\ &\quad = \sum_{n_{N-1}=0}^{\infty} f^{i_1\cdots i_M}(u^1)^{m_1+n_{N-1}}\cdots(u^{N-1})^{m_{N-1}+n_{N-1}}(u^N)^{n_{N-1}}. \end{aligned} \tag{3.12}$$

We use n_{N-1} from (3.8) as the expansion parameter. Each term in (3.12) has a coefficient with a total number of indices M given by (3.10). This coefficient is decomposed into a set of $\mathrm{US}(N)_L$ irreps according to the rule (3.9).

If the charges ($[\mathrm{U}(1)_R]^{N-1}$ projections) of the harmonic function do not appear in the canonical order (3.7), then one should reorder the indices $1, 2, \ldots, N$ so that they can label a Young tableau. For instance, the $N = 4$ function f^{122233} should be rewritten as f^{222331}, so it corresponds to the Young tableau (3, 2, 1). If a complete set of N different projections is present, it can be suppressed, that is, the $N = 4$ function $f^{11234} \equiv f^1$. Finally, if the function carries lower indices (projections of the complex conjugate fundamental representation), they should be converted into sets of $N - 1$ upper indices, for example, the $N = 4$ function $f_4^1 \equiv f^{1123}$ or $f_1^{12} \equiv f^{12234} \equiv f^2$.

3.1.3. Harmonic Derivatives

The harmonic derivatives are operators that respect the defining relations (3.3):

$$\partial_J^I = u_i^I \frac{\partial}{\partial u_i^J} - u^i{}_J \frac{\partial}{\partial u_I^i} - \frac{1}{N} \sum_{K=1}^{N} \delta_J^I \left(u_i^K \frac{\partial}{\partial u_i^K} - u^i{}_K \frac{\partial}{\partial u_K^i} \right). \tag{3.13}$$

They act on the harmonics as follows:

$$\partial_J^I u_i^K = \delta_J^K u_i^I - \frac{1}{N} \delta_J^I u_i^K, \qquad \partial_J^I u^i{}_K = -\partial_K^I u^i{}_J + \frac{1}{N} \delta_J^I u^i{}_K. \tag{3.14}$$

Note that we prefer to treat u_i^I and u_I^i as independent variables subject to the constraints (3.3).

Clearly, the derivatives ∂_J^I are the generators of the group $\mathrm{SU}(N)_R$ acting on the $[\mathrm{U}(1)_R]^{N-1}$ projected indices of the harmonics. The assumption (3.4) is then translated into the requirement that the harmonic functions $f(u)$ are eigenfunctions of the diagonal derivatives ∂_I^I that count the $\mathrm{U}(1)_R$ charges:

$$\left(\partial_I^I - \partial_N^N\right) f_{L_1 \cdots L_r}^{K_1 \cdots K_q}(u) = \left(\delta_{K_1 I} - \delta_{K_1 N} - \delta_{L_1 I} + \delta_{L_1 N} + \cdots\right) f_{L_1 \cdots L_r}^{K_1 \cdots K_q}(u). \tag{3.15}$$

Then the independent harmonic derivatives on the coset are the $N(N-1)/2$ complex derivatives ∂_J^I, $I < J$ corresponding to the raising operators of $\mathrm{SU}(N)_R$ (or their conjugates ∂_J^I, $I > J$ corresponding to the lowering operators of $\mathrm{SU}(N)_R$).

From the preceding discussions, it follows that the harmonic differential conditions

$$\partial_J^I f_{L_1 \cdots L_r}^{K_1 \cdots K_q}(u) = 0, \quad I < J \tag{3.16}$$

impose severe constraints on the harmonic function. Indeed, if the function is of the type (3.7), it is reduced to just one harmonic monomial giving rise to an $\mathrm{SU}(N)$ irrep whose HWS is labeled by the charges. Any other harmonic function subject to the condition (3.16) must vanish.

As an example, take $N = 2$ and the function $f^1(u)$ (3.5) subject to the constraint

$$\partial_2^1 f^1(u) = 0 \Rightarrow f^1(u) = f^i u_i^1 \tag{3.17}$$

since this is the only term in the expansion (3.5) that automatically satisfies the condition (3.17). So, the harmonic function is reduced to a doublet of SU(2). Similarly, for $N = 4$ the function $f^{12}(u)$ is reduced to the $\underline{6}$ of SU(4). Indeed, the constraints $\partial_3^2 f^{12}(u) = \partial_4^3 f^{12}(u) = 0$ ensure that $f^{12}(u)$ depends on u^1 and u^2 only, $f^{12}(u) = f^{ij} u_i^1 u_j^2$. Then the constraint $\partial_2^1 f^{12}(u) = f^{ij} u_i^1 u_j^1 = 0$ implies $f^{ij} = -f^{ji}$. A harmonic function that vanishes if subject to the constraint (3.16) is, for example, in $N = 2$, $f_1(u) \equiv f^2(u)$, since no term in its expansion can satisfy the condition $\partial_2^1 f^2(u) = 0$.

Note that not all of the derivatives ∂_J^I, $I < J$ are independent, as follows from the SU(N) algebra. The independent ones,

$$\partial_2^1, \partial_3^2, \ldots, \partial_N^{N-1} \tag{3.18}$$

correspond to the simple roots of SU(N). Then the constraint (3.16) is equivalent to

$$\partial_{I+1}^I f_{L_1 \cdots L_r}^{K_1 \cdots K_q}(u) = 0, \quad I = 1, \ldots, N-1. \tag{3.19}$$

We remark that the coset SU(N)/U(1)$^{N-1}$ can be parametrized by $N(N-1)/2$ complex coordinates. In our context this amounts to making a choice of the harmonic matrix u_i^I such that the group $[\mathrm{U}(1)_R]^{N-1}$ is identified with $[\mathrm{U}(1)_L]^{N-1} \subset \mathrm{SU}(N)_L$. Then the harmonic derivatives become Cartan's covariant derivatives on the coset. The constraints (3.16) take the form of covariant Cauchy–Riemann analyticity conditions. For this reason we can call the set of constraints (3.16) (or (3.19)) harmonic (H)analyticity conditions. The above argument shows that H analyticity is equivalent to defining a HWS of SU(N), that is, it is the SU(N) irreducibility condition on the harmonic functions.

3.2. (N, p, q) Harmonic Superfields

The main purpose of introducing harmonic variables is to be able to define manifestly SU(N) covariant superfields living in the G-analytic superspaces (2.38). This is done following the example of the chiral superfields. There we replaced the condition (2.11) by the differential chirality constraint (2.14). In the case of (N, p, q) analyticity we have to replace conditions (2.39) by analogous differential constraints. The crucial point now is to let the superfield depend on the harmonic variables and obtain the adequate $[\mathrm{U}(1)]^{N-1}$ projections with the help of harmonic variables:

$$D_\alpha^I \Phi(x, \theta, \bar{\theta}, u) = \bar{D}_J^{\dot{\alpha}} \Phi(x, \theta, \bar{\theta}, u) = 0 \tag{3.20}$$

where

$$D_\alpha^I = D_\alpha^i u_i^I, \qquad \bar{D}_J^{\dot{\alpha}} = \bar{D}_i^{\dot{\alpha}} u_J^i, \quad 1 \le I \le P, \quad N - q + 1 \le J \le N. \tag{3.21}$$

The derivatives appearing in (3.20) anticommute (see (2.16)), therefore there exists a G-analytic basis in superspace,

$$x_A^\mu = x^\mu - i(\theta_1\sigma^\mu\bar\theta^1 + \cdots + \theta_p\sigma^\mu\bar\theta^p - \theta_{N-q+1}\sigma^\mu\bar\theta^{N-q+1} - \cdots - \theta_N\sigma^\mu\bar\theta^N),$$
$$\theta_I^\alpha = \theta_i^\alpha u_I^i, \qquad \bar\theta^{\dot\alpha I} = \bar\theta^{\dot\alpha i} u_i^I, \tag{3.22}$$

where these derivatives become just $D_\alpha^I = \partial/\partial\theta_I^\alpha$, $\bar D_{\dot\alpha J} = -\partial/\partial\bar\theta^{\dot\alpha J}$. Consequently, in this basis the analytic superfield (3.20) becomes an unconstrained function of $N-p$ θs and $N-q$ $\bar\theta$s, as well as of the harmonic variables:

$$\Phi(x_A, \theta_{p+1}, \ldots \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u). \tag{3.23}$$

Let us now turn to the harmonic dependence in (3.23). In principle, each component in the θ expansion of the superfield is a harmonic function having an infinite harmonic expansion of the type (3.12). If we want to deal with a finite set of fields, we have to impose a harmonic irreducibility condition of the type (3.16) (or the equivalent subset (3.19)). However, in the G-analytic basis (3.22) the harmonic derivatives become covariant, D_J^I. In particular, the derivatives

$$D_J^I = \partial_J^I + 2i\theta_J\sigma^\mu\bar\theta^I\partial_\mu - \theta_J\partial^I + \bar\theta^I\bar\partial_J, \quad 1 \le I \le N-q, \ p+1 \le J \le N \tag{3.24}$$

acquire space-time derivative terms. In the next section we shall see that this has important consequences on a G-analytic superfield subject to the additional H analyticity constraints

$$D_J^I\Phi^{[a_1,\ldots,a_{N-1}]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^{N-q}, u) = 0, \quad 1 \le I < J \le N. \tag{3.25}$$

Here we have indicated the SU(N) representation carried by the superfield.

3.3. (N, p, q) Conformal Superfields

So far in this section we have only discussed G-analytic superfields as representations of Poincaré supersymmetry. From the analysis of Section 2 we know that superconformal invariance yields additional restrictions, in particular, on the SU(N) irrep carried by the superfield. Adapting the arguments of Section 2, one finds that (3.20) implies the following harmonic conditions (even if we do not impose the SU(N) irreducibility conditions (3.25)):

$$D_{I+1}^I\Phi^{[a_1,\ldots a_{N-1}]} = D_{I+1}^I\Phi^{[a_1,\ldots,a_{N-1}]} = 0,$$
$$1 \le I \le p-1 \quad \text{and} \quad N-q+1 \le I \le N-1. \tag{3.26}$$

These two subsets of raising and lowering operators of SU(N) generate the algebra of SU(p)$\times$ SU(q). In the spirit of the coset construction of Section 2 this means that we have added the factor SU(p)$\times$ SU(q) to the denominator of the harmonic coset. In other words, a conformally covariant (N, p, q) superfield lives not only

in a smaller superspace, but also in a smaller harmonic space as compared to our initial coset $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-1}$. From Section 2 we also know that the Dynkin labels of such a superfield are restricted (see (2.41)). To summarize, a G-analytic conformal superfield has the form

$$\Phi^{[0,\ldots,0,a_p,\ldots,a_{N-q},0,\ldots,0]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar{\theta}^1, \ldots, \bar{\theta}^{N-q}, u) \tag{3.27}$$

and lives in the harmonic coset

$$\begin{aligned}
&\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p-q+1} \times \mathrm{SU}(p) \times \mathrm{SU}(q)} \quad \text{for } p \geq 2, \quad q \geq 2;\\
&\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q} \times \mathrm{SU}(q)} \quad \text{for } p = 0, 1, \quad q \geq 2;\\
&\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p} \times \mathrm{SU}(p)} \quad \text{for } p \geq 2, \quad q = 0, 1;\\
&\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-1}} \quad \text{for } p = 0, 1 \quad \text{and} \quad q = 0, 1.
\end{aligned} \tag{3.28}$$

This effective reduction of the harmonic coset has been pointed out in Howe and Leeming (1994) and Hartwell and Howe (1995a,b). For example, in the particular case

$$\begin{aligned}
&\Phi^{[0,\ldots,0,a_p,0,\ldots,0,a_{N-q},0,\ldots,0]}(x_A, \theta_{p+1}, \ldots, \theta_N, \bar{\theta}^1, \ldots, \bar{\theta}^{N-q}, u) \Rightarrow\\
&\quad u \in \frac{\mathrm{SU}(N)}{\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q) \times \mathrm{U}(N-p-q))}.
\end{aligned} \tag{3.29}$$

Note that in the limiting cases $N = p + q$ and $N = p + q + 1$ the two cosets (3.28) and (3.29) coincide.

4. MASSLESS SUPERCONFORMAL MULTIPLETS

Massless multiplets are a particular class of superconformal multiplets. Their components are fields carrying Lorentz spin $(j_1, 0)$, $\phi_{\alpha_1\cdots\alpha_{2j_1}}(x)$ or $(0, j_2)$, $\bar{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}(x)$ (all indices are symmetrized). In addition, they satisfy the massless field equations

$$\partial^\mu \sigma_\mu^{\alpha\dot{\alpha}} \phi_{\alpha\alpha_2\cdots\alpha_{2j_1}} = 0, \qquad \partial^\mu \sigma_\mu^{\alpha\dot{\alpha}} \bar{\phi}_{\dot{\alpha}\dot{\alpha}_2\cdots\dot{\alpha}_{2j_2}} = 0 \tag{4.1}$$

(or $\Box\phi = 0$ in the case of spin $(0, 0)$). These massless fields are known (Binegar *et al.*, 1983) to form UIRs of the conformal algebra SU(2,2) if $\ell = j + 1$. Consequently, the massless superconformal multiplets form UIRs of $\mathrm{SU}(2, 2/N)$ (Binegar, 1986; Dobrev and Petkova, 1985, 1987).

In the language of AdS supersymmetry such multiplets are called "supersingletons" (Günaydin and Warner, 1985; Nicolai and Sezgin, 1984).

In this section we shall formulate the massless multiplets of SU(2, 2/N) first in terms of ordinary superfields and then, for a subclass of them, in (N, k, $N-k$) harmonic superspace.[13]

4.1. Massless Multiplets as Constrained Superfields

There exist three types of massless N-extended superconformal multiplets. They can be described in terms of ordinary constrained superfields (Howe *et al.*, 1981; Siegel, 1981).

(i). The first type is given by scalar superfields

$$W^{i_1\cdots i_k}\left(x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}\right), \quad k = 1, \ldots, N-1 \tag{4.2}$$

with k totally antisymmetrized indices of the fundamental representation of SU(N) (i.e., carrying Dynkin labels $[0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0]$). They satisfy the following constraints:

$$D_\alpha^{(j} W^{i_1)i_2\cdots i_k} = 0, \tag{4.3}$$

$$\bar{D}_{\dot{\alpha}\{j} W^{i_1\}i_2\cdots i_k} = 0 \tag{4.4}$$

where ( ) means symmetrization and { } means the traceless part. In the cases $N = 2, 3, 4$ these constraints define the on-shell $N = 2$ matter (hyper)multiplet (Sohnius, 1978) and the $N = 3, 4$ on-shell super-Yang–Mills multiplets (Sohnius, 1978). Their generalization to arbitrary N has been given in Howe *et al.* (1981) and Siegel (1981) where it has also been shown that they describe on-shell massless multiplets.

After rewriting the constraints (4.3) and (4.4) in harmonic superspace in Section 4.2, we shall see that the above massless multiplets are superconformal if

$$\ell = 1, \quad r = \frac{2k}{N} - 1. \tag{4.5}$$

We also note their SU(N) quantum numbers

$$m_1 = \cdots = m_k = 1, \qquad m_{k+1} = \cdots = m_{N-1} = 0, \qquad m = k. \tag{4.6}$$

(ii). The second type is given by a chiral scalar superfield

$$\bar{D}_i^{\dot{\alpha}} \Phi = 0 \tag{4.7}$$

satisfying the additional constraint (field equation)

$$D^{i\alpha} D_\alpha^j \Phi = 0. \tag{4.8}$$

[13] The simplest example is provided by the $N = 2$ hypermultiplet (Galperin *et al.*, 1984); the next example is the $N = 3, 4$ on-shell SYM field-strength (Galperin *et al.*, 1985; Kallosh, 1985; Rosly and Schwarz, 1986; Galperin *et al.*, 1987; Bandos, 1988); the generalization to the case (N, k, $N-k$) was given in Hartwell and Howe (1995a,b).

This superfield is an SU(N) singlet. The corresponding massless multiplet is superconformal if (see Section 2.1)

$$\ell = -r = 1. \tag{4.9}$$

Similarly, one can introduce an antichiral multiplet:

$$D^i_\alpha \bar{\Phi} = 0, \qquad \bar{D}_{i\dot{\alpha}} D^{\dot{\alpha}}_j \bar{\Phi} = 0 \tag{4.10}$$

with quantum numbers

$$\ell = r = 1. \tag{4.11}$$

(iii). The third type is given by chiral superfields carrying external Lorentz spin $(j_1, 0)$:

$$\bar{D}^{\dot{\alpha}}_i w_{\alpha_1 \cdots \alpha_{2j_1}} = 0. \tag{4.12}$$

Here the $2j_1$ spinor indices are totally symmetrized. These superfields are SU(N) singlets. They satisfy the massless field equation

$$D^{i\alpha} w_{\alpha\alpha_2 \cdots \alpha_{2j_1}} = 0. \tag{4.13}$$

As we have seen in Section 2.1, conformal supersymmetry requires that

$$\ell = -r = j_1 + 1. \tag{4.14}$$

Similarly, one can introduce antichiral superfields with Lorentz spin $(0, j_2)$:

$$D^i_\alpha \bar{w}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2j_2}} = 0, \quad \bar{D}^{\dot{\alpha}}_i \bar{w}_{\dot{\alpha}\dot{\alpha}_2 \cdots \dot{\alpha}_{2j_2}} = 0 \tag{4.15}$$

with

$$\ell = r = j_2 + 1. \tag{4.16}$$

It is straightforward to see that such massless representations coincide with the massless supermultiplets of N-extended Pioncaré supersymmetry (for an $N = 8$ example see Günaydin and Marcus (1985).).

4.2. Type (i) Massless Multiplets as Analytic Superfields

Now, let us use the harmonic variables to covariantly project all the SU(N) indices in the constraints (4.3) and (4.4) onto $(\mathrm{U}(1)_R]^{N-1}$. For example, the projection

$$W^{12\cdots k} = W^{i_1 i_2 \cdots i_k}(x, \theta, \bar{\theta}) u^1_{i_1} u^2_{i_2} \cdots u^k_{i_k} \tag{4.17}$$

satisfies the constraints

$$D^1_\alpha W^{12\cdots k} = D^2_\alpha W^{12\cdots k} = \cdots = D^k_\alpha W^{12\cdots k} = 0, \tag{4.18}$$

$$\bar{D}_{\dot{\alpha}\, k+1} W^{12\cdots k} = \bar{D}_{\dot{\alpha}\, k+2} W^{12\cdots k} = \cdots = \bar{D}_{\dot{\alpha}\, N} W^{12\cdots k} = 0 \tag{4.19}$$

where $D^I_\alpha = D^i_\alpha u^I_i$ and $\bar{D}_{\dot\alpha I} = \bar{D}_{\dot\alpha i} u^i_I$. The first of them, Eq. (4.18), is a corollary of the commuting nature of the harmonics variables, and the second one, Eq. (4.19), of the defining conditions (3.3). In Eqs. (4.18) and (4.19) one recognizes the conditions for G analyticity (3.20) of the type $(N, k, N-k)$. As explained in Section 3.2, in the appropriate G-analytic basis (3.22) $W^{12\cdots k}$ becomes an unconstrained function of k $\bar\theta$s and $N-k$ θs.

$$W^{12\cdots k} = W^{12\cdots k}(x_A, \theta_{k+1}, \ldots, \theta_N, \bar\theta^1, \ldots, \bar\theta^k, u). \tag{4.20}$$

It is important to realize that the G-analytic superfield (4.20) is an SU(N) covariant object only because it depends on the harmonic variables. To recover the original harmonic-independent but constrained superfield $W^{i_1 i_2 \cdots i_k}(x, \theta, \bar\theta)$ (4.3), (4.4) we need to impose differential constraints involving the harmonic variables. In Section 3.2 we have shown that they take the form of SU(N) irreducibility conditions, Eq. (3.25). In this particular case they are

$$D^I_j W^{12\cdots k} = 0, \quad 1 \le I < J \le N \tag{4.21}$$

or the equivalent set

$$D^I_{I+1} W^{12\cdots k} = 0, \quad 1 \le I < J \le N-1. \tag{4.22}$$

In the initial real basis (2.7) of the full superspace $\mathbb{R}^{4|2N,2N}$ these constraints simply mean that the superfield is a polynomial in the harmonics, as in (4.17). However, in the G-analytic basis (3.22) the harmonic derivatives (3.24) contain space-time derivatives. This leads to a number of constraints on the component fields. The detailed analysis can be found in (Ferrara and Sokatchev, 2000), here we only recall the final result:

$$\begin{aligned}
W^{12\cdots k} = {} & \phi^{12\cdots k} \\
& + \bar\theta^1_{\dot\alpha} \bar\psi^{\dot\alpha\, 23\cdots k} + \cdots + \bar\theta^k_{\dot\alpha} \bar\psi^{\dot\alpha\, 12\cdots k-1} \\
& + \theta^\alpha_{k+1} \chi_\alpha^{1\cdots k\, k+1} + \cdots + \theta^\alpha_N \chi_\alpha^{1\cdots k\, N} \\
& + \bar\theta^1_{\dot\alpha} \bar\theta^2_{\dot\beta} \bar\psi^{(\dot\alpha\dot\beta)\, 3\cdots k} + \cdots + \bar\theta^{k-1}_{\dot\alpha} \bar\theta^k_{\dot\beta} \bar\psi^{(\dot\alpha\dot\beta)\, 1\cdots k-2} \\
& + \theta^\alpha_{k+1} \theta^\beta_{k+2} \chi_{(\alpha\beta)}^{1\cdots k\, k+1\, k+2} + \cdots + \theta^\alpha_{N-1} \theta^\beta_N \chi_{(\alpha\beta)}^{1\cdots k\, N-1\, N} \\
& \cdots \\
& + \bar\theta^1_{\dot\alpha_1} \cdots \bar\theta^k_{\dot\alpha_k} \bar\psi^{(\dot\alpha_1 \cdots \dot\alpha_k)} + \theta^{\alpha_1}_{k+1} \cdots \theta^{\alpha_{N-k}}_N \chi_{(\alpha_1 \cdots \alpha_{N-k})} \\
& + \text{ derivative terms.}
\end{aligned} \tag{4.23}$$

Here all the component fields belong to totally antisymmetric irreps of SU(N), for example, $\phi^{12\cdots k}(x, u) = \phi^{[i_1 i_2 \cdots i_k]}(x) u^1_{i_1} u^2_{i_2} \cdots u^k_{i_k}$. Further, these fields satisfy massless field equations of the type (4.1).

We conclude this section by a remark concerning the conformal properties of the above multiplets. The $(N, k, N-k)$ analytic superfield $W^{12\cdots k}$ is characterized by the SU(N) quantum numbers $m_1 = \cdots = m_k = 1$, $m_{k+1} = \cdots = m_{N-1} = 0$. From Eqs. (2.40) we see that if

$$\ell_k = 1, \qquad r_k = \frac{2k}{N} - 1 \tag{4.24}$$

$W^{12\cdots k}$ realizes a massless UIR of the superconformal algebra.

5. UIRs OF $D = 4$ N-EXTENDED CONFORMAL SUPERSYMMETRY

In this section we shall show how the complete classification of UIRs of SU(2, 2/N) found in Dobrev and Petkova (1985, 1987) (see also Binegar (1986) and Morel *et al.* (1986) for the massless case) can be obtained by multiplying the three types of massless superfields introduced in Section 4.

5.1. The Three Series of UIRs

The results of Dobrev and Petkova (1985, 1987)[14] fall into three distinct series. The simplest one (called series C in Ferrara and Sokatchev (2000)) is given by the following conditions:

$$\text{C:} \quad \ell = m_1, \qquad r = \frac{2m}{N} - m_1, \; j_1 = j_2 = 0. \tag{5.1}$$

We can construct the superfield realization of series C by multiplying massless G-analytic superfields[15] ("supersingletons") of the type (4.20):

$$W^{[a_1,\ldots,a_{N-1}]} = (W^1)^{a_1}(W^{12})^{a_2}\cdots(W^{12\cdots N-1})^{a_{N-1}}. \tag{5.2}$$

Since each factor in (5.2) satisfies the usual harmonic irreducibility constraints, the same is true for the product:

$$D^I_J W^{[a_1,\ldots,a_{N-1}]} = 0, \quad 1 \le J < I \le N. \tag{5.3}$$

As a result, the lowest component of the superfield (5.2) is an irrep of SU(N) with Dynkin labels $[a_1, \ldots, a_{N-1}]$. This is easily seen by realizing that (i) all the SU(N) indices projected with harmonics u_i^K for a given K are symmetrized; (ii) their total number is $m_K = \sum_{i=K}^{N-1} a_i$; (iii) the harmonic conditions (5.3) remove all symmetrizations between indices projected with different harmonics u_i^K and

[14] Our conventions differ from those of Dobrev and Petkova (1985, 1987) in the following sense: $r \to -r$, $2m/N \to 2m_1 - 2m/N$.

[15] Series of operators obtained as powers of the $N = 4$ super-Yang–Mills field strength considered as a G-analytic harmonic superfield were introduced in Howe and West (1996, 1997, 1999). They were identified with short multiplets of $SU(2, 2/4)$ and their correspondence with the K–K spectrum of IIB supergravity was established in Andrianopoli and Ferrara (1998a,b, 1999).

u_i^L. All this reproduces the structure of a Young tableau with numbers of boxes $(m_1, m_2, \ldots, m_{N-1})$, that is, Dynkin labels $[a_1, \ldots, a_{N-1}]$.

Further, from (4.24) we find $\ell = \sum_{k=1}^{N-1} a_k \ell_k = m_1$ and $r = \sum_{k=1}^{N-1} a_k r_k = \frac{2m}{N} - m_1$, which exactly reproduces (5.1). Thus, we have proved that the complete series C is realized by the product (5.2) of massless multiplets.

We remark that for a generic choice of the Dynkin labels the superfield (5.2) is $(N, 1, 1)$ G-analytic. However, if the first $p-1$ or the last $q-1$ (or both) factors in (5.2) are absent, that is, if the corresponding Dynkin labels vanish, we obtain further analyticity conditions of the type (N, p, q), in accord with (3.27). We should mention that in Dobrev and Petkova (1985, 1987) a list of the possible superconformal differential conditions on superfields is given. There one only finds $(N, 1, 1)$ G analyticity conditions, but this can be explained by the observation made before.

The second series (called B in Ferrara and Sokatchev (2000)) is given by the following conditions:

$$\text{B:} \quad \ell = -r + \frac{2m}{N} \geq 2 + 2j_1 + r + 2m_1 - \frac{2m}{N}, \quad j_2 = 0 \tag{5.4}$$

(or $j_1 \to j_2, r \to -r, \frac{2m}{N} \to 2m_1 - \frac{2m}{N}$). It can be obtained by multiplying the G-analytic massless superfield (5.2) by left-handed chiral ones as follows:

$$w_{\alpha_1 \cdots \alpha_{2j_1}} \Phi^k W^{[a_1, \ldots, a_{N-1}]} \tag{5.5}$$

where $k \geq 0$ is an integer. The first factor in (5.5) brings in the Lorentz spin $(j_1, 0)$. The second factor adjusts the dimension and R charge of the series,

$$\ell = 1 + j_1 + m_1 + k, \qquad r = -1 - j_1 - k - m_1 + \frac{2m}{N}, \tag{5.6}$$

so that they exactly match (5.4). The conformal bound in (5.4) is obtained for $k = 0$, that is, without employing any scalar chiral superfields. The alternative series of this type is obtained by replacing chiral by antichiral superfields.

Finally, the most general series (called A in Ferrara and Sokatchev (2000)) is given by the following conditions:

$$\text{A:} \quad \ell \geq 2 + 2j_2 - r + \frac{2m}{N} \geq 2 + 2j_1 + r + m_1 - \frac{2m}{N} \tag{5.7}$$

(or $j_1 \to j_2, r \to -r, \frac{2m}{N} \to 2m_1 - \frac{2m}{N}$). This series is obtained by multiplying together all possible types of massless superfields:

$$w_{\alpha_1 \cdots \alpha_{2j_1}} \bar{w}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2j_2}} \Phi^k \bar{\Phi}^s W^{[a_1, \ldots, a_{N-1}]} \tag{5.8}$$

where $k \geq s \geq 0$ are integers. This time we find

$$\ell = 2 + j_1 + j_2 + m_1 + k + s, \qquad r = j_2 - j_1 - k + s - m_1 + \frac{2m}{N}, \tag{5.9}$$

which corresponds to (5.7). The two conformal bounds in (5.7) are saturated for $s = 0$ or $k = s = 0$, that is, without employing one or the other type (or both) of scalar chiral superfields. These bounds correspond to superfields satisfying differential constraints, as explained in Section 5.3. The alternative series is obtained by taking $s \geq k \geq 0$.

Note that in the abstract series (5.4) and (5.7) the dimension ℓ and R charge r can be any real numbers. To account for this, the powers k and s in (5.5) and (5.8) will have to take noninteger values, although this might violate unitarity. This does not happen for series C where ℓ is always integer and r is rational.

One final remark concerns the unitarity of the above series of representations. Earlier we mentioned that the massless multiplets (supersingletons) are known to be UIRs of the superconformal algebra. Then it is clear that by multiplying them as we did above we automatically obtain the series of UIRs.

5.2. Series Obtained From One Type of Supersingleton

In Section 5.1 we used all possible G-analytic supersingletons $W^{12\cdots n}$ with $1 \leq n \leq N-1$ to reproduce the complete series C. An alternative approach is to use different realizations of the same type of supersingleton (i.e., for a fixed value of n). We presented a similar construction in Ferrara and Sokatchev (2000), where we only considered the case $n = N/2$ (for even N). The generalization is straightforward. The result is a series of UIRs that is a particular case of the series B above.

The supersingleton $W^{12\cdots n}$ can be equivalently rewritten by choosing different harmonic projections of its SU(N) indices and, consequently, different sets of G analyticity constraints. This amounts to superfields of the type

$$W^{I_1 I_2 \cdots I_n}\left(\theta_{J_{n+1}}, \ldots, \theta_{J_N}, \bar{\theta}^{I_1}, \ldots, \bar{\theta}^{I_n}\right) \tag{5.10}$$

where $I_1, \ldots, I_n$ and $J_{n+1}, \ldots, J_N$ are two complementary sets of N indices. Each of these superfields depends on $2N$ Grassmann variables, that is, half of the total number of $4N$. This is the minimal size of a G-analytic superspace, so we can say that the Ws are the "shortest" superfields (superconformal multiplets).

The idea now is to start multiplying different versions of the Ws of the type (5.10) (for a fixed value of n) in order to obtain composite objects, depending on various numbers of odd variables. The following choice of Ws and of the order of multiplication covers all possible intermediate types of G analyticity:

$$\begin{aligned}
&A(p_1, p_2, \ldots, p_{N-1}) \\
&\quad = [W^{1\cdots n}(\theta_{n+1\cdots N}\bar{\theta}^{1\cdots n})]^{p_1+\cdots+p_{N-1}} \\
&\qquad \times [W^{1\cdots n-1\, n+1}(\theta_{\underline{n}\, n+2\cdots N}\bar{\theta}^{1\cdots n-1\, \underline{n+1}})]^{p_2+\cdots+p_{N-1}} \\
&\qquad \times [W^{1\cdots n-1\, n+2}(\theta_{n\, n+1\, n+3\cdots N}\bar{\theta}^{1\cdots n-1\, \underline{n+2}})]^{p_3+\cdots+p_{N-1}}
\end{aligned}$$

$$
\begin{aligned}
&\cdots \\
&\times [W^{1\cdots n-1\,N-1}(\theta_{n\cdots N-2\,N}\overline{\theta}^{1\cdots n-1\,\underline{N-1}})]^{p_{N-n}+\cdots+p_{N-1}} \\
&\times [W^{1\cdots n-2\,n\,n+1}(\theta_{\underline{n-1}\,n+2\cdots N}\overline{\theta}^{1\cdots n-2\,n\,n+1})]^{p_{N-n+1}+\cdots+p_{N-1}} \\
&\times [W^{1\cdots n-3\,n-1\,n\,n+1}(\theta_{\underline{n-2}\,n+2\cdots N}\overline{\theta}^{1\cdots n-3\,n-1\,n\,n+1})]^{p_{N-n+2}+\cdots+p_{N-1}} \\
&\cdots \\
&\times [W^{13\cdots n+1}(\theta_{\underline{2}\,n+2\cdots N}\overline{\theta}^{13\cdots n+1})]^{p_{N-2}+p_{N-1}} \\
&\times [W^{23\cdots n+1}(\theta_{\underline{1}\,n+2\cdots N}\overline{\theta}^{23\cdots n+1})]^{p_{N-1}}.
\end{aligned}
\tag{5.11}
$$

The power $\sum_{r=k}^{N-1} p_r$ of the kth W is chosen in such a way that each new p_r corresponds to bringing in a new realization of the same supersingleton. As a result, at each step a new θ or $\overline{\theta}$ appears (they are underlined in (5.11)), thus adding new odd dimensions to the G-analytic superspace. The only exception of this rule is the second step at which both a new θ and a new $\overline{\theta}$ appear. So, the series (5.11) covers the cases $(N, n, N-n)$, $(N, n-1, N-n-1)$ and then all intermediate cases up to $(N, 1, 0)$.

The superfield $A(p_1, p_2, \ldots, p_{N-1})$ should be submitted to the same H-analyticity constraints as one would impose on $W^{1\cdots n}$ alone,

$$
D^I_{I+1} A(p_1, p_2, \ldots, p_{N-1}) = 0, \quad I = 1, 2, \ldots, N-1. \tag{5.12}
$$

This is clearly compatible with the G-analyticity conditions on $A(p_1, p_2, \ldots, p_{N-1})$ since they form a subset of these on $W^{1\cdots n}$. As before, H analyticity makes $A(p_1, p_2, \ldots, p_{N-1})$ irreducible under SU(N).

By counting the number of occurrences of each projection $1, 2, \ldots, N-1$ and the dimensions and R charges in (5.11), we easily find the relations

$$
\ell = \sum_{k=1}^{N-1} k p_k, \quad m_1 = \ell - p_{N-1}, \quad m = n\ell, \quad r = \left(\frac{2n}{N} - 1\right)\ell. \tag{5.13}
$$

If $N = 2n$ this series has no R charge. If $p_{N-1} = 0$ the product (5.11) represents a G-analytic superfield and is thus a particular case of the series C. If $p_{N-1} \geq 1$ it depends on all θs and on all $\overline{\theta}$s but $\overline{\theta}^N$, so it is a particular case of the series B (5.6) with $j_1 = 0$.

Finally, the Dynkin labels of the SU(N) irrep carried by the first component of $A(p_1, p_2, \ldots, p_{N-1})$ are given below:

$$
\begin{aligned}
a_1 &= p_{N-2}, \\
a_2 &= p_{N-3}, \ldots, \quad a_{n-2} = p_{N-n+1}, \\
a_{n-1} &= (N-n-2) \sum_{k=N-n+1}^{N-1} p_k + \sum_{k=2}^{N-n} (k-1) p_k,
\end{aligned}
$$

$$
\begin{aligned}
a_n &= p_1, \\
a_{n+1} &= p_2 + \sum_{k=N-n+1}^{N-1} (k - N + n) p_k, \\
a_{n+2} &= p_3, \ldots, \quad a_{N-2} = p_{N-n-1}, \\
a_{N-1} &= \sum_{k=N-n}^{N-1} p_k.
\end{aligned} \tag{5.14}
$$

A particular interesting case is obtained if $a_{N-1} = 0$. This implies $p_{N-n} = \ldots = p_{N-1} = 0$, so $a_1 = \cdots = a_{n-2} = 0$. In other words, this is a G-analytic superfield of the type $(N, n-1, 2)$. The remaining Dynkin labels are $a_{n-1} = \sum_{k=2}^{N-n-1}(k-1)p_k, a_n = p_1, a_{n+1} = p_2, \ldots, a_{N-2} = p_{N-n-1}$. In general, none of these labels vanishes, therefore the harmonic coset in which this $(N, n-1, 2)$ superfield lives is not smaller than the expected one, $\mathrm{SU}(N)/[\mathrm{U}(1)]^{N-n} \times \mathrm{SU}(n-1) \times \mathrm{SU}(2)$.

5.3. Shortness Conditions

In the AdS literature the term "short" applies to multiplets that do not reach their maximal spin (equal to $(j_1 + \frac{N}{2}, j_2 + \frac{N}{2})$ where (j_1, j_2) is the spin of the first component) or which contain constrained fields like, for example, conserved vectors. Our construction of the UIRs of $\mathrm{SU}(2, 2/N)$ in terms of supersingletons allows us to easily find out when and what type of "shortness" condition takes place.

To this end we recall that the building blocks w, Φ and W are all constrained superfields corresponding to the "ultrashort" supersingleton multiplets. They are either G-analytic ((4.18), (4.19)) or chiral ((4.7), (4.12)). In addition, they satisfy on-shell constraints that take the form of $\mathrm{SU}(N)$ irreducibility harmonic conditions (4.21) in the G-analytic case or are of the type (4.8) or (4.13) in the chiral case.

Now, the most general product of chiral, antichiral, and G-analytic superfields as in the series A (5.8) only satisfies the harmonic constraints (4.21) (recall that w and Φ are harmonic-independent). However, there are a number of particular cases where some constraints on the θ dependence still take place.

(i) The product $w_{\alpha_1 \cdots \alpha_{2j_1}} W^{[a_1, \ldots, a_{N-1}]}$ satisfies the intersection of the constraints (4.12), (4.13) of the factor w with the G-analyticity ones of the factor W. In the generic case the latter is of the type $(N, 1, 1)$, so we have

$$
\bar{D}_N^{\dot{\alpha}} \left(w_{\alpha_1 \cdots \alpha_{2j_1}} W^{[a_1, \ldots, a_{N-1}]} \right) = 0, \tag{5.15}
$$

$$
D^{1\alpha} \left(w_{\alpha\alpha_2 \cdots \alpha_{2j_1}} W^{[a_1, \ldots, a_{N-1}]} \right) = 0. \tag{5.16}
$$

If W carries Dynkin labels like in (3.27), it is of the type (N, p, q) and, correspondingly, we obtain q equations like (5.15) and p ones like (5.16).

Similarly, the product $\Phi W^{[\alpha_1,\ldots,\alpha_{N-1}]}$ satisfies the constraints

$$\bar{D}_N^{\dot\alpha}\left(\Phi W^{[a_1,\ldots,a_{N-1}]}\right) = 0, \tag{5.17}$$

$$D^{1\alpha} D_\alpha^1\left(\Phi W^{[a_1,\ldots,a_{N-1}]}\right) = 0 \tag{5.18}$$

or more of the same type in W is (N, p, q) analytic.

(ii) The bilinear products of chiral with antichiral superfields are current-like objects. They satisfy constraints which turn the top spin in the superfield into a conserved "current." The simplest example is the bilinear $\Phi\bar{\Phi}$:

$$D^{i\alpha} D_\alpha^j(\Phi\bar{\Phi}) = 0, \qquad \bar{D}_{i\dot\alpha}\bar{D}_j^{\dot\alpha}(\Phi\bar{\Phi}) = 0. \tag{5.19}$$

These constraints can be weakened if we multiply $\Phi\bar{\Phi}$ by a G-analytic factor W. In this case only certain projections of (5.19) are preserved, for example,

$$D^{1\alpha} D_\alpha^1\left(\Phi\bar{\Phi}W^{[a_1,\ldots,a_{N-1}]}\right) = \bar{D}_{N\dot\alpha}\bar{D}_N^{\dot\alpha}\left(\Phi\bar{\Phi}W^{[a_1,\ldots,a_{N-1}]}\right) = 0. \tag{5.20}$$

Yet another current-like object is the bilinear $w_{\alpha_1\cdots\alpha_{2j_1}}\,\bar{w}_{\dot\alpha_1\cdots\dot\alpha_{2j_2}}$. It satisfies the constraints

$$\bar{D}_i^{\dot\alpha}\left(w_{\alpha_1\cdots\alpha_{2j_1}}\bar{w}_{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2j_2}}\right) = 0, \tag{5.21}$$

$$D^{i\alpha}\left(w_{\alpha\alpha_2\cdots\alpha_{2j_1}}\bar{w}_{\dot\alpha_1\cdots\dot\alpha_{2j_2}}\right) = 0. \tag{5.22}$$

As before, the product $w_{\alpha_1\cdots\alpha_{2j_1}}\,\bar{w}_{\dot\alpha_1\cdots\dot\alpha_{2j_2}}\,W^{[a_1,\ldots,a_{N-1}]}$ satisfies only the corresponding projections of (5.21) and (5.22).

Similarly, the bilinear $w_{\alpha_1\cdots a_{2j_1}}\bar{\Phi}$ satisfies the constraints

$$D^{i\alpha}\left(w_{\alpha\alpha_2\cdots\alpha_{2j_1}}\bar{\Phi}\right) = 0, \tag{5.23}$$

$$\bar{D}_{i\dot\alpha}\bar{D}_j^{\dot\alpha}\left(w_{\alpha_1\cdots\alpha_{2j_1}}\bar{\Phi}\right) = 0. \tag{5.24}$$

(iii) A different class of "short" objects are obtained from the most general product (5.8) of series A either by setting $s = 0$ or $j_2 = 0$ and $s = 1$. In other words, we take the currentlike bilinears above and multiply them by a BPS object (i.e., product of a chiral and a G-analytic factor). The resulting objects satisfy the constraints (for a generic W)

$$\bar{D}_N^{\dot\alpha}\left(w_{\alpha_1\cdots\alpha_{2j_1}}\bar{w}_{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2j_2}}\Phi^k W^{[a_1,\ldots,a_{N-1}]}\right) = 0, \tag{5.25}$$

$$\bar{D}_{N\dot\alpha}\bar{D}_N^{\dot\alpha}\left(w_{\alpha_1\cdots\alpha_{2j_1}}\bar{\Phi}\Phi^k W^{[a_1,\ldots,a_{N-1}]}\right) = 0. \tag{5.26}$$

We call such objects "intermediate short." Note that they saturate the first conformal bound in (5.7). Intermediate short multiplets, as they are defined above, will also occur in $d = 6$ and $d = 3$ (see Sections 6.4 and 7.4).

5.4. BPS States of SU (2, 2/N)

Here we give a summary of the SU$(2, 2/N)$ multiplets that correspond to BPS states.[16] They are realized in terms of superfields that do not depend on at least one spinor coordinate. There are three distinct ways to obtain such multiplets.

5.4.1. (p, q) BPS States

Superfields that do not depend on the first p θs and the last q $\bar{\theta}$s are obtained by multiplying G-analytic objects:

$$\frac{p+q}{2N} \text{ BPS: } W^{[0,\ldots,0,a_p,a_{p+1},\ldots,a_{N-q},0,\ldots,0]}(\theta_{p+1},\ldots,\theta_N,\bar{\theta}^1,\ldots,\bar{\theta}^{N-q})$$
$$= (W^{12\cdots p})^{a_p}(W^{12\cdots p+1})^{a_{p+1}}\cdots(W^{12\cdots N-q})^{a_{N-q}} \tag{5.27}$$

where

$$1 \le p, q \le N-1, \quad p+q \le N. \tag{5.28}$$

Note that the fraction of supersymmetry preserved by a (p, q) BPS state ranges as follows:

$$\frac{1}{N} \le \frac{p+q}{2N} \le \frac{1}{2}. \tag{5.29}$$

The two end points are obtained for $p = q = 1$ and for $p + q = N$.

Such states have the first $p-1$ and the last $q-1$ SU(N) Dynkin labels vanishing. The remaining quantum numbers are

$$\ell = \sum_{k=p}^{N-q} a_k, \qquad j_1 = j_2 = 0, \qquad r = \sum_{k=p}^{N-q}\left(\frac{2k}{N}-1\right)a_k. \tag{5.30}$$

Generically, such superfields live in the harmonic space

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-p-q+1}\times\mathrm{SU}(p)\times\mathrm{SU}(q)}. \tag{5.31}$$

If a subset of the Dynkin labels vanish, for instance,

$$a_{p+m} = a_{p+m+1} = \cdots = a_{N-q-n} = 0, \quad p+q+m+n \le N,$$

the coset (5.31) is further restricted to

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{m+n}\times\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{SU}(N-p-q-m-n+2)}. \tag{5.32}$$

[16] Note that such BPS states have a close resemblance to BPS Poincaré multiplets in five dimensions (Hull, 2000), as expected by a limiting procedure.

5.4.2. (0, q) *BPS States*

Superfields that do not depend on the last q $\bar{\theta}$s (or, alternatively, on the first p θs) are obtained by multiplying G-analytic objects by left- (or right-) handed chiral ones:

$$\frac{q}{2N}\ \text{BPS:}\quad W^{[a_1,a_2,\ldots,a_{N-q},0,\ldots,0]}_{\alpha_1\cdots\alpha_{2j_1}}\big(\theta_1,\ldots,\theta_N,\bar{\theta}^1,\ldots,\bar{\theta}^{N-q}\big)$$
$$= w_{\alpha_1\cdots\alpha_{2j_1}}\Phi^s(W^1)^{a_1}(W^{12})^{a_2}\cdots(W^{12\cdots N-q})^{a_{N-q}} \tag{5.33}$$

where $s \geq 0$ is an integer and

$$1 \leq q \leq N-1. \tag{5.34}$$

Note that the fraction of supersymmetry preserved by a (0, q) BPS state ranges as follows:

$$\frac{1}{2N} \leq \frac{q}{2N} \leq \frac{N-1}{2N}. \tag{5.35}$$

Such states have the last $q-1$ SU(N) Dynkin labels vanishing. The remaining quantum numbers are

$$\ell = 1 + j_1 + s + \sum_{k=p}^{N-q} a_k, \qquad j_2 = 0, \qquad r = -1 - j_1 - s + \sum_{k=p}^{N-q}\left(\frac{2k}{N} - 1\right)a_k. \tag{5.36}$$

Generically, such superfields live in the harmonic space

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q}\times\mathrm{SU}(q)}. \tag{5.37}$$

If a subset of the Dynkin labels vanish, for instance,

$$a_i = 0, \quad 1 \leq n \leq N-q-1,$$

the coset (5.31) is further restricted to

$$\frac{\mathrm{SU}(N)}{[\mathrm{U}(1)]^{N-q-n}\times\mathrm{SU}(q)\times\mathrm{SU}(n+1)}. \tag{5.38}$$

5.4.3. Chiral BPS States

These are described by superfields that do not depend on all of the $\bar{\theta}$s (or, alternatively, on the θs), that is, which are left- (or right-) handed chiral:

$$\frac{1}{2}\ \text{BPS:}\quad W_{\alpha_1\cdots\alpha_{2j_1}}(\theta_1,\ldots,\theta_N) = w_{\alpha_1\cdots\alpha_{2j_1}}\Phi^s. \tag{5.39}$$

They are SU(N) singlets. The remaining quantum numbers are:

$$\ell = 1 + j_1 + s, \qquad j_2 = 0, \qquad r = -1 - j_1 - s. \tag{5.40}$$

The chiral superfields are harmonic-independent.

6. THE SIX-DIMENSIONAL CASE

The method described above can also be applied to the superconformal algebras OSp(8*/2N) in six dimensions. We will first examine the consequences of G-analyticity and conformal supersymmetry and find out the relation to BPS states. Then we will construct UIRs of OSp(8*/2N) by multiplying supersingletons. The results exactly match the general classification of UIRs of OSp(8*/2N) of Minwalla (1998). Some of the results relevant to the cases $N = 1, 2$ have already been presented in Ferrara and Sokatchev (2000).

6.1. The Conformal Superalgebra OSp(8*/2N) and Grassmann Analyticity

The part of the conformal superalgebra OSp(8*/2N) relevant to our discussion is

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 2\Omega^{ij}\gamma^\mu_{\alpha\beta}P_\mu, \tag{6.1}$$

$$\left\{S^{\alpha i}, S^{\beta j}\right\} = 2\Omega^{ij}\gamma^{\alpha\beta}_\mu K^\mu, \tag{6.2}$$

$$\left\{Q^i_\alpha, S^{\beta j}\right\} = i\Omega^{ij}(\gamma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta^\beta_\alpha(4T^{ij} - i\Omega^{ij}D), \tag{6.3}$$

$$\left[D, Q^i_\alpha\right] = \frac{i}{2}Q^i_\alpha, \qquad [D, S^{\alpha i}] = -\frac{i}{2}S^{\alpha i}, \tag{6.4}$$

$$\left[T^{ij}, Q^k_\alpha\right] = -\frac{1}{2}\left(\Omega^{ki}Q^j_\alpha + \Omega^{kj}Q^i_\alpha\right), \tag{6.5}$$

$$[T^{ij}, T^{kl}] = \frac{1}{2}\left(\Omega^{ik}T^{lj} + \Omega^{il}T^{kj} + \Omega^{jk}T^{li} + \Omega^{jl}T^{ki}\right). \tag{6.6}$$

Here Q^i_α are the generators of Poincaré supersymmetry carrying a right-handed chiral spinor index $\alpha = 1, 2, 3, 4$ of the Lorentz group SU*(4) $\sim$ SO(5, 1) (generators $M_{\mu\nu}$) and an index $i = 1, 2, \ldots, 2N$ of the fundamental representation of the R symmetry group USp($2N$) (generators $T^{ij} = T^{ji}$); $S^{\beta j}$ are the generators of conformal supersymmetry carrying a left-handed chiral spinor index; D is the generator of dilations, P_μ of translations and K_μ of conformal boosts. It is convenient to make the nonstandard choice of the symplectic matrix $\Omega^{ij} = -\Omega^{ji}$ with nonvanishing entries $\Omega^{1\,2N} = \Omega^{2\,2N-1} = \cdots = \Omega^{N\,N+1} = 1$. The chiral spinors satisfy a pseudoreality condition of the type $\overline{Q^i_\alpha} = \Omega^{ij}Q^\beta_j c_{\beta\alpha}$ where c is a 4×4 unitary "charge conjugation" matrix. Note that the generators M, P, K, D form the lie algebra of SO(8*) $\sim$ SO(2, 6) and the generators Q and S form an SO(8*) chiral spinor.

The standard realization of this superalgebra makes use of the superspace

$$\mathbb{R}^{6|8N} = \frac{\mathrm{OSp}(8^*/2N)}{\{K, S, M, D, T\}} = (x^\mu, \theta^{\alpha i}) \tag{6.7}$$

where $\theta^{\alpha i}$ is a left-handed spinor. Unlike the four-dimensional case, here chirality is not an option but is already built in. The only way to obtain smaller superspaces is through Grassmann analyticity. We begin by imposing a single condition of G-analyticity (cf. Eq. (2.20)):

$$q^1_\alpha \Phi(x, \theta) = 0, \tag{6.8}$$

which amounts to considering the coset

$$\mathbb{A}^{6|4(2N-1)} = \frac{\mathrm{OSp}(8^*/2N)}{\{K, S, M, D, T, Q^1\}} = (x^\mu, \theta^{\alpha 1,2,\ldots,2N-1}) \tag{6.9}$$

(note that with our conventions $\theta^{\alpha 1} = \theta^\alpha_{2N}, \ldots, \theta^{\alpha N} = \theta^\alpha_{N+1}, \theta^{\alpha\, N+1} = -\theta^\alpha_n \ldots, \theta^{\alpha\, 2N} = -\theta^\alpha_1$). From the algebra (6.1)–(6.6) we obtain

$$m_{\mu\nu} = 0, \tag{6.10}$$

$$t^{1\,1} = t^{1\,2} = \cdots = t^{1\,2N-1} = 0, \tag{6.11}$$

$$4t^{1\,2N} + \ell = 0. \tag{6.12}$$

Equation (6.10) implies that the superfield Φ must be a Lorentz scalar. To interpret Eqs. (6.11) and (6.12), we need to split the generators of USp($2N$) into raising operators (corresponding to the positive roots):

$$T^{k\,2N-l}, \qquad k = 1, \ldots, N, \qquad l = k, \ldots, 2N-k \text{ (simple if } l = k); \tag{6.13}$$

$[\mathrm{U}(1)]^N$ charges:

$$H_k = -2T^{k\,2N-k+1}, \qquad k = 1, \ldots, N; \tag{6.14}$$

the remaining generators are lowering operators (corresponding to the negative roots). The Dynkin labels a_k of a USp($2N$) irrep are defined as follows:

$$a_k = H_k - H_{k+1}, \qquad k = 1, \ldots, N-1, \qquad a_N = H_N, \tag{6.15}$$

so that, for instance, the generator Q^1 is the HWS of the fundamental irrep $(1, 0, \ldots, 0)$.

Now it becomes clear that (6.11) is part of the USp($2N$) irreducibility conditions, whereas (6.12) relates the conformal dimension to the sum of the Dynkin labels:

$$\ell = 2\sum_{k=1}^{N} a_k. \tag{6.16}$$

Let us denote the highest-weight UIRs of the OSp(8*/$2N$) algebra by

$$\mathcal{D}(\ell; J_1, J_2, J_3; a_1, \ldots, a_N)$$

where ℓ is the conformal dimension, J_1, J_2, J_3 are the SU*(4) Dynkin labels and a_k are the USp($2N$) Dynkin labels of the first component. Then the G-analytic

superfields defined previously are of the type

$$\Phi(\theta^{1,2,\ldots,2N-1}) \Leftrightarrow \mathcal{D}\left(2\sum_{k=1}^{N} a_k; 0, 0, 0; a_1, \ldots, a_N\right). \tag{6.17}$$

The next step is to add the generator Q^2_α to the superspace coset denominator:

$$\mathbb{A}^{6|4(2N-2)} = \frac{\mathrm{OSp}(8^*/2N)}{\{K, S, M, D, T, Q^1, Q^2\}} = (x^\mu, \theta^{\alpha 1,2,\ldots,2N-2}). \tag{6.18}$$

This implies the new constraints

$$4t^{2\,2N-1} + \ell = 0 \Rightarrow a_1 = 0, \tag{6.19}$$

$$t^{2\,2N} = 0. \tag{6.20}$$

Note that the vanishing of the lowering operator $t^{2\,2N}$ means that the subalgebra $\mathrm{SU}(2) \subset \mathrm{USp}(2N)$ formed by $t^{1\,2N-1}$, $t^{2\,2N}$ and $t^{1\,2N}$, $-t^{2\,2N-1}$ acts trivially on the particular $\mathrm{USp}(2N)$ irreps. This is equivalent to setting $a_1 = 0$, as in (6.19). Thus, the new G-analytic superfields are of the type

$$\Phi(\theta^{1,2,\ldots,2N-2}) \Leftrightarrow \mathcal{D}\left(2\sum_{k=2}^{N} a_k; 0, 0, 0; 0, a_2, \ldots, a_N\right). \tag{6.21}$$

From (6.1) it is clear that we can go on in the same manner until we remove half of the θs, namely $\theta^{N+1}, \ldots, \theta^{2N}$. Each time we have to set a new Dynkin label to zero. We can summarize by saying that the superconformal algebra $\mathrm{OSp}(8^*/2N)$ admits the following short UIRs corresponding to BPS states:

$$\frac{p}{2N}\ \text{BPS:}\quad \mathcal{D}\left(2\sum_{k=p}^{N} a_k; 0, 0, 0; 0, \ldots, 0, a_p, \ldots, a_n\right), \quad p = 1, \ldots, N. \tag{6.22}$$

6.2. Supersingletons

There exist three types of massless multiplets in six dimensions corresponding to ultrashort UIRs (supersingletons) of $\mathrm{OSp}(8^*/2N)$ (see, e.g., Günaydin and Takemae (1999) for the case $N = 2$). All of them can be formulated in terms of constrained superfields as follows.

(i) The first type is described by a superfield $W^{\{i_1\cdots i_n\}}(x, \theta)$, $1 \le n \le N$, which is antisymmetric and traceless in the external $\mathrm{USp}\,(2N)$ indices (for even n one can impose a reality condition). It satisfies the constraint (see Howe *et al.* (1983) and Park (1999)).

$$D^{(k}_\alpha W^{\{i_1)i_2\cdots i_n\}} = 0 \ \Rightarrow \mathcal{D}(2; 0, 0, 0; 0, \ldots, 0, a_n = 1, 0, \ldots, 0), \tag{6.23}$$

where the spinor covariant derivatives obey the supersymmetry algebra

$$\left\{D^i_\alpha, D^j_\beta\right\} = -2i\Omega^{ij}\gamma^\mu_{\alpha\beta}\partial_\mu. \tag{6.24}$$

The components of this superfield are massless fields. In the case $N = n = 1$ this is the on-shell (1, 0) hypermultiplet and for $N = n = 2$ it is the on-shell (2, 0) tensor multiplet (Howe *et al.*, 1983; Bergshoeff *et al.*, 1986).

(ii) The second type is described by a (real) superfield without external indices, $w(x, \theta)$ obeying the constraint

$$D^{(i}_{[\alpha} D^{j)}_{\beta]} w = 0 \;\Rightarrow \mathcal{D}(2; 0, 0, 0; 0, \ldots, 0). \tag{6.25}$$

(iii) Finally, there exists an infinite series of multiplets described by superfields with n totally symmetrized external Lorentz spinor indices, $w_{(\alpha_1\cdots\alpha_n)}(x, \theta)$ (they can be made real in the case of even n) obeying the constraint

$$D^i_{[\beta} w_{(\alpha_1]\cdots\alpha_n)} = 0 \;\Rightarrow \mathcal{D}(2 + n/2; n, 0, 0; 0, \ldots, 0). \tag{6.26}$$

As shown in Ferrara and Sokatchev (2000), the six-dimensional massless conformal fields only carry reps $(J_1, 0)$ of the little group SU(2) $\times$ SU(2) of a lightlike particle momentum. This result is related to the analysis of conformal fields in d dimensions (Siegel, 1989; Angelopoulos and Laoues, 1998). This fact implies that massless superconformal multiplets are classified by a single SU(2) and USp $(2N)$ R symmetry and are therefore identical to massless super-Poincaré multiplets in five dimensions. Some physical implication of the above circumstance have recently been discussed in Hull (2000), where it was suggested that certain strongly coupled $d = 5$ theories effectively become six-dimensional.

6.3. Harmonic Superspace

The massless multiplets (i) and (ii) admit an alternative formulation in harmonic superspace (see Howe *et al.* (1985), Zupnik (1986a,b), and Howe (1999) for $N = 1, 2$). The advantage of this formulation is that the constraints (6.23) become conditions for G-analyticity. We introduce harmonic variables describing the coset $\mathrm{USp}(2N)/[\mathrm{U}(1)]^N$:

$$u \in \mathrm{USp}(2N): \quad u^I_i u^i_J = \delta^I_J, \qquad u^I_i \Omega^{ij} u^J_j = \Omega^{IJ}, \qquad u^I_i = (u^i_I)^*. \tag{6.27}$$

Here the indices i and j belong to the fundamental representation of $\mathrm{USp}(2N)$ and I and J are labels corresponding to the $[\mathrm{U}(1)]^N$ projections. The harmonic derivatives

$$D^{IJ} = \Omega^{K(I} u^{J)}_i \frac{\partial}{\partial u^k_i} \tag{6.28}$$

form the algebra of $\mathrm{USp}(2N)_R$ (see (6.6)) realized on the indices I and J of the harmonics.

Let us now project the defining constraint (6.23) with the harmonics $u^K_k u^1_{i_1} \cdots u^n_{i_n}$, $K = 1, \ldots, n$:

$$D^1_\alpha W^{12\cdots n} = D^2_\alpha W^{12\cdots n} = \cdots = D^n_\alpha W^{12\cdots n} = 0 \tag{6.29}$$

where $D^K_\alpha = D^i_\alpha u^K_i$ and $W^{12\cdots n} = W^{\{i_1\cdots i_n\}} u^1_{i_1} \cdots u^n_{i_n}$. Indeed, the constraint (6.23) now takes the form of a G-analyticity condition. In the appropriate basis in superspace the solution to (6.29) is a short superfield depending on part of the odd coordinates:

$$W^{12\cdots n}(x_A, \theta^1, \theta^2, \ldots, \theta^{2N-n}, u). \tag{6.30}$$

In addition to (6.29), the projected superfield $W^{12\cdots n}$ automatically satisfies the USp($2N$) harmonic irreducibility conditions

$$D^{K2N-K} W^{12} = 0, \qquad K = 1, \ldots, N \tag{6.31}$$

(only the simple roots of USp($2N$) are shown). The equivalence between the two forms of the constraint follows from the obvious properties of the harmonic products $u^K_{[k} u^K_{i]} = 0$ and $\Omega^{ij} u^K_i u^L_j = 0$ for $1 \le K < L \le n$. The harmonic constraints (6.31) make the superfield ultrashort.

Finally, in case (ii), projecting the constraint (6.25) with $u^I_i u^I_j$, where $I = 1, \ldots, N$ (no summation), we obtain the condition

$$D^I_\alpha D^I_\beta w = 0. \tag{6.32}$$

It implies that the superfield w is *linear* in each projection $\theta^{\alpha I}$.

6.4. Series of UIRs of OSp(8*/2N) and Shortening

It is now clear that we can realize the BPS series of UIRs (6.22) as products of the different G-analytic superfields (supersingletons) (6.29).[17] BPS shortening is obtained by setting the first $p - 1$ USp($2N$) Dynkin labels to zero:

$$\frac{p}{2N} \text{ BPS:} \quad W^{[0,\ldots,0,a_p,\ldots,a_N]}(\theta^1, \theta^2, \ldots, \theta^{2N-p}) = (W^{1\cdots p})^{a_p} \cdots (W^{1\cdots N})^{a_N} \tag{6.33}$$

(note that even if $a_1 \neq 0$ we still have $1/2N$ shortening).

We remark that our harmonic coset USp($2N$)/[U(1)]^N is effectively reduced to

$$\frac{\mathrm{USp}(2N)}{\mathrm{U}(p) \times [\mathrm{U}(1)]^{N-p}} \tag{6.34}$$

in the case of $p/2N$ BPS shortening (just as it happened in four dimensions). Such a smaller harmonic space was used in (Howe, 1999) to formulate the (2, 0) tensor multiplet.

[17] As a bonus, we also prove the unitarity of these series, since they are obtained by multiplying massless unitary multiplets.

A study of the most general UIRs of OSp(8*/2N) (similar to the one of Dobrev and Petkova (1985, 1987) for the case of SU(2, 2/N) is presented in Minwalla (1998). We can construct these UIRs by multiplying the three types of supersingletons, (6.33)–(6.35).

$$w_{\alpha_1\cdots\alpha_{m_1}} w_{\beta_1\cdots\beta_{m_2}} w_{\gamma_1\cdots\gamma_{m_3}} w^k W^{[a_1,\ldots,a_N]} \tag{6.35}$$

where $m_1 \geq m_2 \geq m_3$ and the spinor indices are arranged so that they form an SU*(4) UIR with Young tableau (m_1, m_2, m_3) or Dynkin labels $J_1 = m_1 - m_2$, $J_2 = m_2 - m_3$, $J_3 = m_3$. Thus we obtain four distinct series:

$$\begin{aligned}
&\text{A)} \quad \ell \geq 6 + \frac{1}{2}(J_1 + 2J_2 + 3J_3) + 2\sum_{k=1}^{N} a_k;\\
&\text{B)} \quad J_3 = 0, \quad \ell \geq 4 + \frac{1}{2}(J_1 + 2J_2) + 2\sum_{k=1}^{N} a_k;\\
&\text{C)} \quad J_2 = J_3 = 0, \quad \ell \geq 2 + \frac{1}{2}J_1 + 2\sum_{k=1}^{N} a_k;\\
&\text{D)} \quad J_1 = J_2 = J_3 = 0, \quad \ell = 2\sum_{k=1}^{N} a_k.
\end{aligned} \tag{6.36}$$

The superconformal bound is saturated when $k = 0$ in (6.35). Note that the values of the conformal dimension we can obtain are "quantized" since the factor w^k has $\ell = 2k$ and k must be a nonnegative integer to ensure unitarity. With this restriction Eq. (6.36) reproduces the results of Minwalla (1998). However, we cannot comment on the existence of a "window" of dimensions $2 + \frac{1}{2}J_1 + 2\sum_{k=1}^{N} a_k \leq \ell \leq 4 + \frac{1}{2}J_1 + 2\sum_{k=1}^{N} a_k$ conjectured in Minwalla (1998).[18]

In the generic case the multiplet (6.35) is "long," but for certain special values of the dimension some shortening can take place (Minwalla, 1998). We can immediately identify all these short multiplets. First of all, case D corresponds to BPS shortening. In the other cases let us first set $a_i = 0$, that is, no BPS multiplets appear in (6.35). Then saturating the bound in case A (i.e, setting $k = 0$) leads to the shortening condition (see (6.26)):

$$\epsilon^{\delta\alpha\beta\gamma} D^i_\delta \big(w_{\alpha\cdots\alpha_{m_1}} w_{\beta\cdots\beta_{m_2}} w_{\gamma\cdots\gamma_{m_3}}\big) = 0 \to \ell = 6 + \frac{1}{2}(J_1 + 2J_2 + 3J_3). \tag{6.37}$$

Next, in case B we have two possibilities: either we saturate the bound ($k = 0$) or

[18] In a recent paper (Ferrara and Fronsdal,) the UIRs of the six-dimensional conformal algebra SO(2,6) have been classified. Note that the superconformal bound in case A (with all $a_i = 0$) is stronger that the purely conformal unitarity bounds found in Ferrara and Fronsdal (2000).

we use just one factor, w ($k = 1$). Using (6.25) and (6.26), we find

$$\epsilon^{\delta\gamma\alpha\beta} D^i_\gamma \left(w_{\alpha\cdots\alpha_{m_1}} w_{\beta\cdots\beta_{m_2}}\right) = 0 \to \ell = 4 + \frac{1}{2}(J_1 + 2J_2); \tag{6.38}$$

$$\epsilon^{\delta\gamma\alpha\beta} D^{(i}_\delta D^{j)}_\gamma \left(w w_{\alpha\cdots\alpha_{m_1}} w_{\beta\cdots\beta_{m_2}}\right) = 0 \to \ell = 6 + \frac{1}{2}(J_1 + 2J_2). \tag{6.39}$$

Similarly, in case C with $J_1 \neq 0$ we have three options, namely, setting $k = 0 \to \ell = 2 + \frac{1}{2}J_1$ (which corresponds to the supersingleton defining constraint (6.26) or $k = 1, 2$, which gives:

$$\epsilon^{\delta\gamma\beta\alpha} D^{(i}_\gamma D^{j)}_\beta \left(w w_{\alpha\cdots\alpha_{m_1}}\right) = 0 \to \ell = 4 + \frac{1}{2} J_1. \tag{6.40}$$

$$\epsilon^{\delta\gamma\beta\alpha} D^{(i}_\delta D^j_\gamma D^{k)}_\beta \left(w^2 w_{\alpha\cdots\alpha_{m_1}}\right) = 0 \to \ell = 6 + \frac{1}{2} J_1. \tag{6.41}$$

Finally, in case C with $J_1 = 0$ we can take the scalar supersingleton (6.25) itself, that is, set $k = 1 \to \ell = 2$, or set $k = 2, 3$:

$$\epsilon^{\delta\gamma\beta\alpha} D^{(i}_\gamma D^j_\beta D^{k)}_\alpha (w^2) = 0 \to \ell = 4, \tag{6.42}$$

$$\epsilon^{\delta\gamma\beta\alpha} D^{(i}_\delta D^j_\gamma D^k_\beta D^{l)}_\alpha (w^3) = 0 \to \ell = 6. \tag{6.43}$$

Introducing USp($2N$) quantum numbers into these shortening conditions is achieved by multiplying the short multiplets by a BPS object. The new short multiplets satisfy the corresponding USp($2N$) projections of Eqs. (6.25), (6.26), (6.37)–(6.43). We call such objects "intermediate short."

7. THE THREE-DIMENSIONAL CASE

In this section we carry out the analysis of the $d = 3$, $N = 8$ superconformal algebra OSp($8/4, \mathbb{R}$) in a way similar to the above. Some of the results have already been presented in Ferrara and Sokatchev (2000). As in the previous cases, our results could easily be extended to OSp ($N/4, \mathbb{R}$) superalgebras with arbitrary N. The $N = 2$ and $N = 3$ cases were considered in Fabbri *et al*. (2000) and Fré *et al*. (1999).

7.1. The Conformal Superalgebra OSp($8/4, \mathbb{R}$) and Grassmann Analyticity

The part of the conformal superalgebra OSp($8/4, \mathbb{R}$) relevant to our discussion is given below:

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 2\delta^{ij} \gamma^\mu_{\alpha\beta} P_\mu, \tag{7.1}$$

$$\{Q^i_\alpha, S^j_\beta\} = \delta^{ij} M_{\alpha\beta} + 2\epsilon_{\alpha\beta}(T^{ij} + \delta^{ij} D), \tag{7.2}$$

$$[T^{ij}, Q^k_\alpha] = i(\delta^{ki} Q^j_\alpha - \delta^{kj} Q^i_\alpha), \tag{7.3}$$

$$[T^{ij}, T^{kl}] = i(\delta^{ik} T^{jl} + \delta^{jl} T^{ik} - \delta^{jk} T^{il} - \delta^{il} T^{jk}). \tag{7.4}$$

Here we find the following generators: Q^i_α of $N = 8$ Poincaré supersymmetry carrying a spinor index $\alpha = 1, 2$ of the $d = 3$ Lorentz group SL $(2, \mathbb{R}) \sim$ SO $(1, 2)$ (generators $M_{\alpha\beta} = M_{\beta\alpha}$) and a vector[19] index $i = 1, \ldots, 8$ of the R symmetry group SO(8) (generators $T^{ij} = -T^{ji}$); S^i_α of conformal supersymmetry; P_μ, $\mu = 0, 1, 2$, of translations; D of dilations.

The standard realization of this superalgebra makes use of the superspace

$$\mathbb{R}^{3|16} = \frac{\mathrm{OSp}(8/4, \mathbb{R})}{\{K, S, M, D, T\}} = (x^\mu, \theta^{\alpha i}). \tag{7.5}$$

To study G analyticity we need to decompose the generators Q^i_α under $[\mathrm{U}(1)]^4 \subset$ SO(8). Besides the vector representation 8_v of SO(8) we are also going to use the spinor ones, 8_s and 8_c. In this context we find it convenient to introduce the four subgroups U(1) by successive reductions: SO(8) $\to$ SO(2) $\times$ SO(6) $\sim$ U(1) $\times$ SU(4) $\to [\mathrm{SO}(2)]^2 \times \mathrm{SO}(4) \sim [\mathrm{U}(1)]^2 \times \mathrm{SU}(2) \times \mathrm{SU}(2) \to [\mathrm{SO}(2)]^4 \sim [\mathrm{U}(1)]^4$. Denoting the four U(1) charges by $\pm$, $(\pm)$, $[\pm]$, and $\{\pm\}$, we decompose the 3 eight-dimensional representations as follows:

$$8_v\colon\ \ Q^i \to Q^{\pm\pm}, Q^{(\pm\pm)}, Q^{[\pm]\{\pm\}}, \tag{7.6}$$

$$8_s\colon\ \ \phi^a \to \phi^{+(+)[\pm]}, \phi^{-(-)[\pm]}, \phi^{+(-)\{\pm\}}, \phi^{-(+)\{\pm\}} \tag{7.7}$$

$$8_c\colon\ \ \sigma^{\dot a} \to \sigma^{+(+)\{\pm\}}, \sigma^{-(-)\{\pm\}}, \sigma^{+(-)[\pm]}, \sigma^{-(+)[\pm]} \tag{7.8}$$

The definition of the charge operators H_i, $i = 1, 2, 3, 4$, can be read off from the corresponding projections of the relation (7.2):

$$\{Q^{++}_\alpha, S^{--}_\beta\} = \frac{1}{2} M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2} H_1\right),$$

$$\{Q^{(++)}_\alpha, S^{(--)}_\beta\} = \frac{1}{2} M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2} H_2\right),$$

$$\{Q^{[+]\{+\}}_\alpha, S^{[-]\{-\}}_\beta\} = \frac{1}{2} M_{\alpha\beta} + \epsilon_{\alpha\beta}\left(D - \frac{1}{2} H_3 - \frac{1}{2} H_4\right),$$

$$\{Q^{[+]\{-\}}_\alpha, S^{[-]\{+\}}_\beta\} = -\frac{1}{2} M_{\alpha\beta} - \epsilon_{\alpha\beta}\left(D - \frac{1}{2} H_3 + \frac{1}{2} H_4\right). \tag{7.9}$$

[19] Since SO(8) has 3 eight-dimensional representations, 8_v, 8_s, and 8_c related by triality, the choice which one to ascribe to the supersymmetry generators is purely conventional. To be consistent with the other N-extended $d = 3$ supersymmetries where the odd generators always belong to the vector representation, we prefer to put an 8_v index i on the supercharges.

In this notation we have

$$\begin{aligned}\left[H_1,\, Q_\alpha^{\pm\pm}\right] &= \left[H_2,\, Q_\alpha^{(\pm\pm)}\right] = \pm 2i\, Q_\alpha^{\pm\pm},\\ \left[H_3,\, Q^{[\pm]\{\pm\}}\right] &= \left[H_4,\, Q^{[\pm]\{\pm\}}\right] = \pm i\, Q^{[\pm]\{\pm\}}.\end{aligned} \tag{7.10}$$

Let us denote a quasi-primary superconformal field of the OSp(8/4, $\mathbb{R}$) algebra by the quantum numbers of its HWS:

$$\mathcal{D}(\ell;\, J;\, a_1,\, a_2,\, a_3,\, a_4) \tag{7.11}$$

where ℓ is the conformal dimension, J is the Lorentz spin and a_i are the Dynkin labels (see, e.g., Frappat *et al*. (2000)) of the SO(8) R symmetry. In fact, in our scheme the natural labels are the four charges h_i (the eigenvalues of H_i). They are related to the Dynkin labels as follows:

$$\begin{aligned} h_1 &= 2(a_1 + a_2) + a_3 + a_4,\\ h_2 &= 2a_2 + a_3 + a_4,\\ h_3 &= a_3, \quad h_4 = a_4, \end{aligned} \tag{7.12}$$

or, inversely,

$$a_1 = \frac{1}{2}(h_1 - h_2), \quad a_2 = \frac{1}{2}(h_2 - h_3 - h_4), \quad a_3 = h_3, \quad a_4 = h_4. \tag{7.13}$$

A HWS $|a_i\rangle$ of SO(8) is by definition annihilated by the positive simple roots of the SO(8) algebra:

$$T^{[++]}|a_i\rangle = T^{\{++\}}|a_i\rangle = T^{++(--)}|a_i\rangle = T^{(++)[-]\{-\}}|a_i\rangle = 0. \tag{7.14}$$

To build G analytic superspaces we have to add one or more projections of Q_α^i to the coset denominator. In choosing the subset of projections we have to make sure that (i) they anticommute among themselves and (ii) the subset is closed under the action of the raising operators of SO(8) (7.14). Then we have to examine the consistency of the vanishing of the chosen projections with the conformal superalgebra (7.9). Thus we find the following sequence of G analytic superspaces corresponding to BPS states:

$$\frac{1}{8}\ \text{BPS:}\quad \begin{cases} q_\alpha^{++}\Phi = 0 \rightarrow \\ \Phi(\theta^{++},\, \theta^{(\pm\pm)},\, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_1 + a_2 + \frac{1}{2}(a_3 + a_4);\, 0;\, a_1,\, a_2,\, a_3,\, a_4) \end{cases} \tag{7.15}$$

$$\frac{1}{4}\ \text{BPS:}\quad \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = 0 \rightarrow \\ \Phi(\theta^{++},\, \theta^{(++)},\, \theta^{[\pm]\{\pm\}}) \\ \mathcal{D}(a_2 + \frac{1}{2}(a_3 + a_4);\, 0;\, 0,\, a_2,\, a_3,\, a_4) \end{cases} \tag{7.16}$$

$$\frac{3}{8}\text{ BPS:}\quad \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[+]\{+\}}\Phi = 0 \to \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, \theta^{[-]\{+\}}) \\ \mathcal{D}(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4) \end{cases} \tag{7.17}$$

$$\frac{1}{2}\text{ BPS (type I):}\quad \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[+]\{\pm\}}\Phi = 0 \to \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}) \\ \mathcal{D}(\frac{1}{2}(a_3; 0; 0, 0, a_3, 0) \end{cases} \tag{7.18}$$

$$\frac{1}{2}\text{ BPS (type II):}\quad \begin{cases} q_\alpha^{++}\Phi = q_\alpha^{(++)}\Phi = q_\alpha^{[\pm]\{+\}}\Phi = 0 \to \\ \Phi(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}) \\ \mathcal{D}(\frac{1}{2}(a_4; 0; 0, 0, 0, a_4) \end{cases} \tag{7.19}$$

Note the existence of two types of $1/2$ BPS states because of the two possible subsets of projections of q^i closed under the raising operators of SO(8) (7.14).

We remark that in the cases $1/4$, $3/8$, and $1/2$, the states are annihilated by some of the lowering operators of SO(8). This means that certain subalgebras of SO(8) act trivially on them:

$$\frac{1}{4}:\quad \text{SU (2)} \leftrightarrow \{T^{++(--)}, T^{--(++)}, H_1 - H_2 \tag{7.20}$$

$$\frac{3}{8}:\quad \text{SU (3)} \leftrightarrow \begin{cases} T^{++(--)}, T^{--(++)}, H_1 - H_2 \\ T^{(++)[-]\{-\}}, T^{(--)[+]\{+\}}, H_2 - H_3 - H_4 \end{cases} \tag{7.21}$$

$$\frac{1}{2}:\quad \text{SU (4)}_I \leftrightarrow \begin{cases} T^{++(--)}, T^{--(++)}, H_1 - H_2 \\ T^{(++)[-]\{-\}}, T^{(--)[+]\{+\}}, H_2 - H_3 - H_4 \\ T^{\{++\}}, T^{\{--\}}, H_4 \end{cases} \tag{7.22}$$

$$\frac{1}{2}:\quad \text{SU (4)}_{II} \leftrightarrow \begin{cases} T^{++(--)}, T^{--(++)}, H_1 - H_2 \\ T^{(++)[-]\{-\}}, T^{(--)[+]\{+\}}, H_2 - H_3 - H_4 \\ T^{[++]}, T^{[--]}, H_3 \end{cases} \tag{7.23}$$

These properties are equivalent to the restrictions on the possible values of the SO(8) Dynkin labels in (7.15)–(7.19). Note that the existence of two types of $1/2$ BPS states can be equivalently explained by the two possible ways to embed SU(4) in SO(8), as shown in (7.22) and (7.23).

7.2. Supersingletons and Harmonic Superspace

The supersingletons are the simplest OSp(8/4, $\mathbb{R}$) representations of the type (7.18) or (7.19) and correspond to $\mathcal{D}(1/2; 0; 0, 0, 1, 0)$ or $\mathcal{D}(1/2; 0; 0, 0, 0, 1)$. The existence of two distinct types of $d = 3$ $N = 8$ supersingletons has first been noted in Günaydin *et al.* (1986). Each of them is just a collection of eight Dirac supermultiplets (Fronsdal, 1982) made out of "Di" and "Rac" singletons (Flato and Fronsdal, 1978, 1980, 1981, 1986).

To realize the supersingletons in superspace we note that the HWS in the two supermultiplets above has spin 0 and the Dynkin labels of the 8_s or 8_c of SO(8), correspondingly. Therefore we take a scalar superfield $\Phi_a(x^\mu, \theta_i^\alpha)$ (or $\sum_{\dot a}(x^\mu, \theta_i^\alpha)$) carrying an external 8_s index a (or an 8_c index $\dot a$). These superfields are subject to the following on-shell constraints[20]:

$$\text{Type I:}\quad D_\alpha^i \Phi_a = \frac{1}{8}\gamma^i_{a\dot b}\tilde\gamma^j_{\dot b c} D_\alpha^j \Phi_c; \tag{7.24}$$

$$\text{Type II:}\quad D_\alpha^i \Sigma_{\dot a} = \frac{1}{8}\tilde\gamma^i_{\dot a b}\gamma^j_{b\dot c} D_\alpha^j \Sigma_{\dot c}. \tag{7.25}$$

The two multiplets consist of a massless scalar in the 8_s (8_c) and spinor in the 8_c (8_s).

The harmonic superspace description of these supersingletons can be realized by taking the harmonic coset[21]

$$\frac{\mathrm{SO}(8)}{[\mathrm{SO}(2)]^4} \sim \frac{\mathrm{Spin}(8)}{[U(1)]^4}. \tag{7.26}$$

Since SO(8) $\sim$ Spin(8) has three inequivalent fundamental representations, 8_s, 8_c, 8_v, following Galperin *et al*. (1992) we introduce three sets of harmonic variables:

$$u_a^A,\ w_{\dot a}^{\dot A},\ v_i^I \tag{7.27}$$

where A, $\dot A$, and I denote the decompositions of an 8_s, 8_c, and 8_v index, correspondingly, into sets of four U(1) charges (see (7.6)–(7.8)). Each of the 8×8 real matrices (7.27) belongs to the corresponding representation of SO(8) $\sim$ Spin(8). This implies that they are orthogonal matrices (this is a peculiarity of SO(8) because of triality):

$$u_a^A u_a^B = \delta^{AB}, \qquad w_{\dot a}^{\dot A} w_{\dot a}^{\dot B} = \delta^{\dot A \dot B}, \qquad v_i^I v_i^J = \delta^{IJ}. \tag{7.28}$$

These matrices supply three copies of the group space, and we only need one to parametrize the harmonic coset. The condition that identifies the three sets of

[20] See also Howe (1999) for the description of a supersingleton related to ours by SO(8) triality. Superfield representations of other $OSp(N/4)$ superalgebras have been considered in Ivanov and Sorin (1980) and Fabbri *et al*. (2000).

[21] A formulation of the above multiplet in harmonic superspace has been proposed in Howe (1999) (see also Zupnik and Khetselius (1988) and Howe and Leeming (1994) for a general discussion of three-dimensional harmonic superspaces). The harmonic coset used in Howe (1999) is Spin (8)/$U(4)$. Although the supersingleton itself does indeed live in this smaller coset (see Section 7.5.4), its residual symmetry $U(4)$ would not allow us to multiply different realizations of the supersingleton. For this reason we prefer from the very beginning to use the coset (7.26) with a minimal residual symmetry.

harmonic variables is[22]

$$u_a^A(\gamma^I)_{A\dot{A}}w_{\dot{a}}^{\dot{A}} = v_i^I(\gamma^i)_{a\dot{a}}. \tag{7.29}$$

Further, we introduce harmonic derivatives (the covariant derivatives on the coset (7.26))

$$D^{IJ} = u_a^A(\gamma^{IJ})^{AB}\frac{\partial}{\partial u_a^B} + w_{\dot{a}}^{\dot{A}}(\gamma^{IJ})^{\dot{A}\dot{B}}\frac{\partial}{\partial w_{\dot{a}}^{\dot{B}}} + v_i^{[I}\frac{\partial}{\partial v_i^{J]}}. \tag{7.30}$$

They respect the algebraic relations (7.28) and (7.29) among the harmonic variables and form the algebra of SO(8), realized on the indices, A, $\dot{A}$, I of the harmonics.

We now use the harmonic variables for projecting the supersingleton defining constraints (7.24) and (7.25). Using the relation (7.29) it is easy to show that the projections $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ satisfy the following G-analyticity constraints:

$$D^{++}\Phi^{+(+)[+]} = D^{(++)}\Phi^{+(+)[+]} = D^{[+]\{\pm\}}\Phi^{+(+)[+]} = 0, \tag{7.31}$$

$$D^{++}\Sigma^{+(+)\{+\}} = D^{(++)}\Sigma^{+(+)\{+\}} = D^{[+]\{\pm\}}\Sigma^{+(+)\{+\}} = 0 \tag{7.32}$$

where $D_\alpha^I = v_i^I D_\alpha^i$, $\Phi^A = u_a^A\Phi_a$ and $\Sigma^{\dot{A}} = w_{\dot{a}}^{\dot{A}}\Sigma_{\dot{a}}$. This is the superspace realization of the 1/2 BPS shortening conditions (7.18) and (7.19). In the appropriate basis in superspace $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ depend on different halves of the odd variables as well as on the harmonic variables:

$$\text{Type I:}\quad \Phi^{+(+)[+]}\big(x_A, \theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}, u, w\big) \tag{7.33}$$

$$\text{Type II:}\quad \Sigma^{+(+)\{+\}}\big(x_A, \theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}, u, w\big). \tag{7.34}$$

In addition to the G analyticity constraints (7.31) and (7.32), the on-shell superfields $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ are subject to the SO(8) irreducibility harmonic conditions obtained from (7.14) by replacing the SO(8) generators by the corresponding harmonic derivatives. The combination of the latter with Eq. (7.31) is equivalent to the original constraint (7.24).

It should be stressed that $\Phi^{+(+)[+]}$ and $\Sigma^{+(+)\{+\}}$ automatically satisfy additional harmonic constraints involving lowering operators of SO(8) (cf. (7.22) and (7.23)). As mentioned earlier, this means that the supersingleton harmonic superfields effectively live in the smaller harmonic coset Spin(8)/U(4).

7.3. OSp(8/4, ℝ) Supersingleton Composites

One way to obtain short multiplets of OSp(8/4, ℝ) is to multiply different analytic superfields describing the Type I supersingleton. The point is that previously we chose a particular projection of, for example, the defining constraint

[22] Although each of the three sets of harmonic variables depends on the same 28 parameters, we need at least two sets to be able to reproduce all possible representations of SO(8).

(7.24) that lead to the analytic superfield $\Phi^{+(+)[+]}$. We could have done this in a variety of ways, each time obtaining superfields depending on different halves of the total number of odd variables. Leaving out the 8_v lowest weight θ^{--}, we can have four distinct but equivalent analytic descriptions of the Type I supersingleton:

$$\begin{aligned}
&\Phi^{+(+)[+]}\big(\theta^{++}, \theta^{(++)}, \theta^{[+]\{+\}}, \theta^{[+]\{-\}}\big),\\
&\Phi^{+(+)[-]}\big(\theta^{++}, \theta^{(++)}, \theta^{[-]\{+\}}, \theta^{[-]\{-\}}\big),\\
&\Phi^{+(-)\{+\}}\big(\theta^{++}, \theta^{(--)}, \theta^{[+]\{+\}}, \theta^{[-]\{+\}}\big),\\
&\Phi^{+(-)\{-\}}\big(\theta^{++}, \theta^{(--)}, \theta^{[+]\{-\}}, \theta^{[-]\{-\}}\big),
\end{aligned} \tag{7.35}$$

Then we can multiply them in the following way:

$$\big(\Phi^{+(+)[+]}\big)^{p+q+r+s}\big(\Phi^{+(+)[-]}\big)^{q+r+s}\big(\Phi^{+(-)\{+\}}\big)^{r+s}\big(\Phi^{+(-)\{-\}}\big)^{s}, \tag{7.36}$$

thus obtaining three series of OSp(8/4, $\mathbb{R}$) UIRs exhibiting 1/8, 1/4, or 1/2 BPS shortening:

$$\begin{aligned}
&\frac{1}{8}\text{BPS:}\quad \mathcal{D}\left(a_1 + a_2 + \frac{1}{2}(a_3 + a_4), 0; a_1, a_2, a_3, a_4\right), \quad a_1 - a_4 = 2s \geq 0;\\
&\frac{1}{4}\text{BPS:}\quad \mathcal{D}\left(a_2 + \frac{1}{2}a_3, 0; 0, a_2, a_3, 0\right);\\
&\frac{1}{2}\text{BPS:}\quad \mathcal{D}\left(\frac{1}{2}a_3, 0; 0, 0, a_3, 0\right)
\end{aligned} \tag{7.37}$$

where

$$a_1 = r + 2s, \qquad a_2 = q, \qquad a_3 = p, \qquad a_4 = r. \tag{7.38}$$

We see that multiplying only one type of supersingletons cannot reproduce the general result of Section 7.1 for all possible short multiplets. Most notably, in (7.37) there is no 3/8 series. The latter can be obtained by mixing the two types of supersingletons:

$$\big[\Phi^{+(+)[+]}\big(\theta^{++}, \theta^{(++)}, \theta^{[+]\{\pm\}}\big)\big]^{a_3}\big[\Sigma^{+(+)\{+\}}\big(\theta^{++}, \theta^{(++)}, \theta^{[\pm]\{+\}}\big)\big]^{a_4} \tag{7.39}$$

(or the same with Φ and Σ exchanged). Counting the charges and the dimension, we find exact matching with the series (7.17):

$$\frac{3}{8}\text{BPS:}\quad \mathcal{D}\left(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4\right). \tag{7.40}$$

Further, mixing two realizations of Type I and one of Type II supersingletons, we can construct the 1/4 series

$$\big[\Phi^{+(+)[+]}\big]^{a_2+a_3}\big[\Phi^{+(+)[-]}\big]^{a_2}\big[\Sigma^{+(+)\{+\}}\big]^{a_4}, \tag{7.41}$$

which corresponds to (7.16):

$$\frac{1}{4}\ \text{BPS:}\quad \mathcal{D}\left(a_2+\frac{1}{2}(a_3+a_4);0;0,a_2,a_3,a_4\right). \tag{7.42}$$

Finally, the full 1/8 series (7.15) (i.e., without the restriction $a_1 - a_4 = 2s \geq 0$ in (7.37)) can be obtained in a variety of ways.

In this section we have analyzed all short highest-weight UIRs of the OSp(8/4, $\mathbb{R}$) superalgebra whose HWSs are annihilated by part of the super-Poincaré odd generators. The number of distinct possibilities have been shown to correspond to different BPS conditions on the HWS. When the algebra is interpreted on the AdS_4 bulk, for which the 3d superconformal field theory corresponds to the boundary M-2 brane dynamics, these states appear as BPS massive excitations, such as K–K states or AdS black holes, of M-theory on $AdS_4 \times S^7$. Since in M-theory there is only one type of supersingleton related to the M-2 brane transverse coordinates (Duff, 1987, 1988), according to our analysis massive states cannot be 3/8 BPS saturated, exactly as it happens in M-theory on $M^4 \times T^7$. Indeed, the missing solution was also noticed in Duff and Liu (1999) by studying AdS_4 black holes in gauged $N = 8$ supergravity. Curiously, in the ungauged theory, which is in some sense the flat limit of the former, the 3/8 BPS states are forbidden (Ferrara and Günaydin, 1998; Ferrara and Maldacena, 1998) by the underlying $E_{7(7)}$ symmetry of $N = 8$ supergravity (Cremmer and Julia, 1979).

7.4. Series of UIRs of OSp(8/4, $\mathbb{R}$)

In the cases of even dimension $d = 4, 6$, we had supersingleton superfields carrying either R symmetry indices, Lorentz indices, or just conformal dimension. Multiplying them we were able to reproduce the corresponding general series of UIRs. In the case of $d = 3$, the situation is different, since we have only two supersingletons carrying SO(8) spinor indices. Multiplying them we could construct the short objects of the BPS type considered above. Yet, for reproducing the most general UIRs (see Minwalla (1998)), we need short objects with spin but without SO(8) indices. These arise in the form of conserved currents. The simplest one is a Lorentz scalar and an SO(8) singlet w of dimension $\ell = 1$. It can be realized as a bilinear of two supersingletons of the same type, for example, $w = \Phi_a \Phi_a$ or $w = \Sigma_{\dot{a}} \Sigma_{\dot{a}}$. Using (7.24) or (7.25) one can show that it satisfies the constraint (a non-BPS shortness condition)

$$D^i_\alpha D^{j\alpha} w = \frac{1}{8}\delta^{ij} D^k_\alpha D^{ka} w. \tag{7.43}$$

The other currents carry SL(2, $\mathbb{R}$) spinor indices, $w_{\alpha_1\cdots\alpha_{2j}}$, have dimension $\ell = 1 + J$ and satisfy the constraint (Park, 1999)

$$D^{i\alpha} w_{\alpha\alpha_2\cdots\alpha_{2J}} = 0. \tag{7.44}$$

They can be constructed as bilinears of the two types of supersingletons (for half-integer spin) or of two copies of the same type (for integer spin). For example, the two lowest ones ($J = 1/2$ and $J = 1$) are

$$w_\alpha = \gamma^i_{b\dot{b}}\big(D^i_\alpha \Phi_b \Sigma_{\dot{b}} - \Phi_b D^i_\alpha \Sigma_{\dot{b}}\big), \tag{7.45}$$

$$w_{\alpha\beta} = D^i_{(\alpha}\Phi_a(\gamma^i\gamma^j)_{ab}D^j_{\beta)}\Phi'_b + 32i(\Phi_a\partial_{\alpha\beta}\Phi'_a - \partial_{\alpha\beta}\Phi_a\Phi'_a). \tag{7.46}$$

They are easily generalized to

$$\begin{aligned} w_{\alpha_1\cdots\alpha_{2n+1}} &= \gamma^i_{b\dot{b}}\sum_{k=0}^{n}(-1)^k \\ &\big(\partial_{(\alpha_1\alpha_2}\cdots\partial_{\alpha_{2k-1}\alpha_{2k}}D^i_{\alpha_{2k+1}}\Phi_b\partial_{\alpha_{2k+2}\alpha_{2k+3}}\cdots\partial_{\alpha_{2n-1}\alpha_{2n})}\Sigma_{\dot{b}} \\ &\quad - \partial_{(\alpha_1\alpha_2}\cdots\partial_{\alpha_{2k-1}\alpha_{2k}}\Phi_b\partial_{\alpha_{2k+1}\alpha_{2k+2}}\cdots\partial_{\alpha_{2n-1}\alpha_{2n}}D^i_{\alpha_{2n+1})}\Sigma_{\dot{b}}\big); \end{aligned} \tag{7.47}$$

$$\begin{aligned} w_{\alpha_1\cdots\alpha_{2n}} &= \sum_{k=0}^{n}(-1)^k \\ &\big[\partial_{(\alpha_1\alpha_2}\cdots\partial_{\alpha_{2k-1}\alpha_{2k}}D^i_{\alpha_{2k+1}}\Phi_a(\gamma^i\gamma^j)_{ab}D^j_{\alpha_{2k+2}}\partial_{\alpha_{2k+3}\alpha_{2k+4}}\cdots\partial_{\alpha_{2n-1}\alpha_{2n})}\Phi'_b \\ &\quad + 32i\,\partial_{(\alpha_1\alpha_2}\cdots\partial_{\alpha_{2k-1}\alpha_{2k}}\Phi_a\partial_{\alpha_{2k+1}\alpha_{2k+2}}\cdots\partial_{\alpha_{2n-1}\alpha_{2n})}\Phi'_a\big] \end{aligned} \tag{7.48}$$

(note that if $n = 2m$ the two supersingletons Φ_a and Φ'_a can be identical).

The generic "long" UIR of OSp(8/4, $\mathbb{R}$) can now be obtained as a product of all of the above short objects:

$$w_{\alpha_1\cdots\alpha_{2j}}w^k\mathrm{BPS}[a_1, a_2, a_3, a_4]. \tag{7.49}$$

Here we have used the first factor to obtain the spin, the second one for the conformal dimension and the BPS factor for the SO(8) quantum numbers. The unitarity bound is given by

$$\ell \geq 1 + J + a_1 + a_2 + \frac{1}{2}(a_3 + a_4) \tag{7.50}$$

and is saturated if $k = 0$ in (7.49). The object (7.49) is short if (i) $J \neq 0$ and $k = 0$ (then it satisfies the intersection of (7.44) with the BPS conditions); (ii) $J = 0$ and $k = 1$ (then it satisfies the intersection of (7.43) with the BPS conditions); (iii) $J = 0$ and $k = 0$ (then it is BPS short). These results exactly match the classification of Minwalla (1998).

7.5. BPS States of OSp(8/4, $\mathbb{R}$)

Here we give a summary of all possible OSp(8/4, $\mathbb{R}$) BPS multiplets. Denoting the UIRs by

$$\mathcal{D}(\ell; J; a_1, a_2, a_3, a_4), \tag{7.51}$$

where ℓ is the conformal dimension, J is the spin, and a_1, a_2, a_3, a_4 are the SO(8) Dynkin labels, we find four BPS conditions:

7.5.1.

$$\frac{1}{8}\text{BPS:} \quad q_\alpha^{++} = 0. \tag{7.52}$$

The corresponding UIRs are

$$\mathcal{D}\left(a_1 + a_2 + \frac{1}{2}(a_3 + a_4); 0; a_1, a_2, a_3, a_4\right) \tag{7.53}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^4}. \tag{7.54}$$

If $a_2 = a_3 = a_4 = 0$ this coset becomes Spin(8)/U(4).

7.5.2.

$$\frac{1}{4}\text{BPS:} \quad q_\alpha^{++} = q_\alpha^{(++)} = 0. \tag{7.55}$$

The corresponding UIRs are

$$\mathcal{D}\left(a_2 + \frac{1}{2}(a_3 + a_4); 0; 0, a_2, a_3, a_4\right) \tag{7.56}$$

and the harmonic coset is

$$\frac{\text{Spin}(8)}{[\text{U}(1)]^2 \times \text{U}(2)}. \tag{7.57}$$

If $a_3 = a_4 = 0$ this coset becomes Spin(8)/U(1) $\times$ [SU(2)]^3.

7.5.3.

$$\frac{3}{8}\text{BPS:} \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = 0. \tag{7.58}$$

The corresponding UIRs are

$$\mathcal{D}\left(\frac{1}{2}(a_3 + a_4); 0; 0, 0, a_3, a_4\right) \tag{7.59}$$

and the harmonic coset is

$$\frac{\mathrm{Spin}(8)}{\mathrm{U}(1) \times \mathrm{U}(3)}. \tag{7.60}$$

7.5.4.

$$\frac{1}{2}\text{BPS (type I):} \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[+]\{\pm\}} = 0; \tag{7.61}$$

$$\frac{1}{2}\text{BPS (type II):} \quad q_\alpha^{++} = q_\alpha^{(++)} = q_\alpha^{[+]\{+\}} = q_\alpha^{[\pm]\{+\}} = 0. \tag{7.62}$$

The corresponding UIRs are

$$\frac{1}{2}\text{BPS (type I):} \quad \mathcal{D}\left(\frac{1}{2}a_3; 0; 0, 0, a_3, 0\right); \tag{7.63}$$

$$\frac{1}{2}\text{BPS (type II):} \quad \mathcal{D}\left(\frac{1}{2}a_4; 0; 0, 0, 0, a_4\right). \tag{7.64}$$

and the harmonic coset is

$$\frac{\mathrm{Spin}(8)}{\mathrm{U}(4)}. \tag{7.65}$$

8. CONCLUSIONS

Here we give a summary of the different types of BPS states that are realized as products of supersingletons described by G-analytic harmonic superfields. We shall restrict ourselves to the physically interesting cases of D3, M_2, and M_5 branes horizon geometry where only one type of such supersingletons appears. This construction gives rise to a restricted class of the most general BPS states.

8.1. PSU (2, 2/4)

The BPS states are constructed in terms of the $N = 4, d = 4$ super-Yang-Mills multiplet W^{ij} in three equivalent G analytic realizations:

$$(W^{12}(\theta_{3,4}, \overline{\theta}^{1,2}))^{p+q+r}(W^{13}(\theta_{2,4}, \overline{\theta}^{1,3}))^{q+r}(W^{23}(\theta_{1,4}, \overline{\theta}^{2,3}))^{r}. \tag{8.1}$$

BPS	SU(4)	Dimension	Harmonic space
$\frac{1}{2}$	(0, p, 0)	p	SU(4)/S(U(2) × U(2))
$\frac{1}{4}$	(q, p, q)	p + 2q	$SU(4)/[U(1)]^3$
$\frac{1}{8}$	(q, p, q + 2r)	p + 2q + 3r	$SU(4)/[U(1)]^3$
	(0, p, 2r)	p + 3r	SU(4)/U(1) × U(2)
	(0, 0, 2r)	3r	SU(4)/U(3)

8.2. OSp(8*/4)

The BPS states are constructed in terms of the (2, 0) $d = 6$ tensor multiplet $W^{\{ij\}}$ in two equivalent G-analytic realizations:

$$(W^{12}(\theta^{1,2})^{p+q}(W^{13}(\theta^{1,3}))^q. \tag{8.2}$$

BPS	USp(4)	Dimension	Harmonic space
$\frac{1}{2}$	(0, p)	2p	USp(4)/U(2)
$\frac{1}{4}$	(2q, p)	2p + 4q	$USp(4)/[U(1)]^2$
	(2q, 0)	4q	USp(4)/U(2)

8.3. OSp(8/4, ℝ)

The Type I BPS states are constructed in terms of the $N = 8$, $d = 3$ matter multiplet Φ_a carrying an external 8_s $SO(8)$ spinor index in four equivalent G-analytic realizations:

$$\begin{aligned}
&\left[\Phi^{+(+)[+]}\left(\theta^{++,(++),[+]\{\pm\}}\right)\right]^{p+q+r+s} \times \\
&\left[\Phi^{+(+)[-]}\left(\theta^{++,(++),[-]\{\pm\}}\right)\right]^{q+r+s} \times \\
&\left[\Phi^{+(-)\{+\}}\left(\theta^{++,(--),[\pm]\{+\}}\right)\right]^{r+s} \times \\
&\left[\Phi^{+(-)\{-\}}\left(\theta^{++,(--),[\pm]\{-\}}\right)\right]^{s}.
\end{aligned} \tag{8.3}$$

BPS	SO(8)	Dimension	Harmonic space
$\frac{1}{2}$	(0, 0, p, 0)	$\frac{1}{2}p$	Spin(8)/U(4)
$\frac{1}{4}$	(0, q, p, 0)	$\frac{1}{2}(p+2q)$	Spin(8)/U(2) × U(2)
$\frac{1}{8}$	(r + 2s, q, p, r)	$\frac{1}{2}(p+2q+3r+4s)$	Spin(8)/[U(1)]^4

The Type II BPS states are constructed in terms of the $N = 8$, $d = 3$ matter multiplet $\Sigma_{\dot{a}}$ carrying an external 8_c $SO(8)$ spinor index in four equivalent G analytic realizations:

$$\begin{aligned}
&\left[\Sigma^{+(+)\{+\}}\left(\theta^{++,(++),[\pm]\{+\}}\right)\right]^{p+q+r+s} \times \\
&\left[\Sigma^{+(+)\{-\}}\left(\theta^{++,(++),[\pm]\{-\}}\right)\right]^{q+r+s} \times \\
&\left[\Sigma^{+(-)[+]}\left(\theta^{++,(--),[+]\{\pm\}}\right)\right]^{r+s} \times \\
&\left[\Sigma^{+(-)[-]}\left(\theta^{++,(--),[-]\{\pm\}}\right)\right]^{s}.
\end{aligned} \tag{8.4}$$

BPS	SO(8)	Dimension	Harmonic space
$\frac{1}{2}$	(0, 0, 0, p)	$\frac{1}{2}p$	Spin(8)/U(4)
$\frac{1}{4}$	(0, q, 0, p)	$\frac{1}{2}(p+2q)$	Spin(8)/U(2) × U(2)
$\frac{1}{8}$	(r+2s, q, r, p)	$\frac{1}{2}(p+2q+3r+4s)$	Spin(8)/[U(1)]^4

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